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Renormalisation of $\mathcal{N} = \frac{1}{2}$ Supersymmetric Theories

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Abstract

A deformation of standard supersymmetric theories is introduced in the form of non-anticommutative superspace coordinates. The properties of four such deformed theories are investigated with a view to exploring the range of theories which may be deformed in a consistent manner. Emphasis is placed on the renormalisation procedure in each case and on finding actions which are form-invariant under renormalisation. Firstly, a deformation of the two-dimensional $\mathcal{N} = 2$ non-linear σ-model is explored and found to be unrenormalisable. Attention then turns to the four-dimensional Wess-Zumino model and its extension to Abelian and non-Abelian gauge theories, and the forms that such actions may take are considered.

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Chapter 1

Introduction

Particle physics is described in terms of the powerful formalism of quantum field theory. The basic premise of this formalism is that elementary particles are considered to be excitations of a continuous set of quantum harmonic oscillators labelled by space-time coordinates [1–3]. Quantum field theory, and especially the standard model of particle physics, agree remarkably well with experiment [4, 5], providing accurate predictions for the interactions of elementary particles through the fundamental forces of electromagnetism, and the weak and strong nuclear forces. However, the continuously infinite degrees of freedom introduced by the formalism give rise to the problem of infinite results for calculations. These infinities are of little consequence in a theory of free (non-interacting) fields as a simple redefinition of the vacuum sets the infinite contribution to the fields' energy at zero. Probability amplitudes for the behaviour of free fields become finite relative to this redefined vacuum. However, with the introduction of interacting fields, matters become rather more complicated and the calculation of finite amplitudes requires a more careful approach. The process of renormalisation is a means to remove unwanted divergences from calculations, and the procedures involved along with their implications will be discussed in chapter 4.

Renormalisation does introduce some new problems, however, which require attention. The solution of one of these, the hierarchy problem, was the motivation for much of the early work on supersymmetry, a possible source of "beyond the standard model" physics which will be introduced in chapter 2. In addition, one of the predictions of renormalisation is that the values of the parameters of a theory will change with the energy scale (or the inverse length scale) at which the theory is defined. If the coupling strength of an interaction increases with energy, then it is found through perturbative calculations to diverge at a finite value known as the Landau pole [6]. The fact that these couplings diverge at large but finite energies is generally taken as evidence that quantum field theory (or at least the standard model) is in fact only an effective theory for low energies and that there is some more fundamental theory that will become apparent at high energies. One suggestion for this theory is that the cutoff appearing in the renormalisation procedure has a physical interpretation in terms of a fundamental "graininess" of spacetime leading to the field of non-commutative geometry [7–9]. It is the combination of these two ideas, non-commutativity and supersymmetry, that gave rise to the study of non-anticommutative models which will be introduced in chapter 3. The remaining chapters of this work will each be concerned with a different example of a non-anticommutative theory and, in particular, how each theory behaves under renormalisation.

Chapter 2

Supersymmetry

2.1 A History of Supersymmetry

Arguably, one of the most compelling attributes of the Standard Model of particle physics is its high degree of symmetry. Indeed, the model is almost entirely characterised by its internal symmetries: $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

A relativistic quantum field theory is also symmetrical under the spacetime symmetries of the Poincaré group. In the 1960's a number of "no-go theorems" were derived which forbade the embedding of the internal symmetries into a larger symmetry group with the Poincaré group in anything but a trivial fashion. The most comprehensive of these theorems is the Coleman-Mandula theorem [10] which states that, subject to certain assumptions, the most general symmetry group that a quantum field theory may possess takes the form of a direct product of the internal group and the Poincaré group (or the conformal group in the case of massless fields). One of the assumptions of the theorem is that all symmetries involved are bosonic in nature; that is, all group generators are defined in terms of their commutators. However, if this assumption is relaxed to allow fermionic generators (defined by their anticommutators) then the Coleman-Mandula theorem may be sidestepped. These fermionic symmetries are known as supersymmetries.

The first example of a model with a fermonic symmetry arose in string theory with the supersymmetry existing between bosonic and fermionic excitations on the two-dimensional world-sheet. This was proposed first by Gervais and Sakita [11] in 1971 and again independently by Neveu and Schwarz [12]. Four-dimensional models were later discovered independently with different motivations. In 1971, Gol'fand and Likhtman explored supersymmetry as a possible extension of the Poincaré group [13]; in 1973, Volkov and Akulov introduced a symmetry between bosons and fermions in the hope that the masslessness of neutrinos might be explained by a symmetry linking the particle to the Goldstone boson of an unbroken symmetry [14]; and also in 1973, Wess and Zumino introduced the model that bears their names as the four-dimensional analog of the world-sheet supersymmetry [15].

It has been shown that the supersymmetry algebra is the only non-trivial extension of the symmetry allowed by the Coleman-Mandula theorem when this assumption is relaxed. This result is known as the Haag-Łopuszański-Sohnius theorem [16]. Shortly after the concepts of supersymmetry were advanced, the construction of supersymmetric Lagrangians was greatly simplified by the work of Salam and Strathdee [17]. In this they reformulated supersymmetry in terms of a higher dimensional superspace consisting of the ordinary coordinates of spacetime along with anticommuting (Grassmann) coordinates.

2.2 The Supersymmetry Algebra

The supersymmetry algebra may be derived [18] from the Poincaré group algebra along with the assumptions of the Coleman-Mandula theorem extended to fermionic generators. The generators of the Poincaré group obey the following relations:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \end{aligned} \tag{2.1}$$

where P_μ is the generator of translations, $M_{\mu\nu}$ is the rotation generator and $\eta_{\mu\nu}$ is the Minkowski metric.

In addition, a quantum field theory with an internal symmetry group has generators which obey the following:

$$\begin{aligned} [R^a, R^b] &= iC^{abc}R^c, \\ [R^a, P_\mu] &= 0, \\ [R^a, M_{\mu\nu}] &= 0, \end{aligned} \tag{2.2}$$

where the C^{abc} are constants related to the internal symmetry group and are equal to the structure constants f^{abc} for a simple group.

The supersymmetry algebra is the most general extension of these relations consistent with the Haag-Łopuszański-Sohnius theorem and its form is completely determined by physicality criteria up to the values of the anticommutators of two spinorial generators of the same representation (known as central charges). In the algebra's most common incarnation, the central charges are taken to be zero, in which case the algebra is given by the following relations (in addition to those of Eqs. (2.1), (2.2)):

$$\begin{aligned} \{Q^i_\alpha, Q^j_\beta\} &= \{\overline{Q}^i_{\dot\alpha}, \overline{Q}^j_{\dot\beta}\} = 0, \\ \{Q^i_\alpha, \overline{Q}^j_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{ij}, \\ \\ [Q^i_\alpha, P_\mu] &= [\overline{Q}^i_{\dot\beta}, P_\mu] = 0, \\ [Q^i_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta, \\ [(\overline{Q}^{\dot\alpha})^i, M_{\mu\nu}] &= (\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\overline{Q}^{\dot\beta})^i, \end{aligned} \tag{2.3}$$

where μ, ν, ρ, σ are Lorentz indices; α, β, $\dot\alpha$, $\dot\beta$ are two-component spinor indices; and $i, j = 1 \ldots \mathcal{N}$ label the supersymmetry generators.

Here, $\sigma^{\mu\nu}, \overline{\sigma}^{\mu\nu}$ are given by

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{1}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu), \\ \overline{\sigma}^{\mu\nu} &= \frac{1}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu). \end{aligned} \tag{2.4}$$

Irreducible representations of the Poincaré group have a definite mass and spin (or helicity in the massless case) due to the fact that the the operators P^2 and W^2 are Casimirs of the group, where W^μ is the Pauli-Lubanski pseudovector given by

$$W^\mu = \epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma. \tag{2.5}$$

In the case of the super-Poincaré group, however, W^2 is no longer a Casimir [19] and as such, the irreducible representations will have states with different spins. Indeed, any irreducible representation will have an equal number of bosonic and fermionic degrees of freedom with equal mass, and this must hold both on- and off-shell. A (Weyl) spinor field has two degrees of freedom on-shell, corresponding to the particle and anti-particle states, and can thus be balanced by a single complex scalar. Off-shell, however, a spinor field has four degree of freedom. It is necessary, therefore, to introduce a second, non-propagating, complex scalar field. This is known as an auxiliary field and, as it does not propagate, it may be eliminated in the on-shell case by means of its equation of motion, allowing for an equal number of bosonic and fermionic degrees of freedom in both cases. With this in mind, we may write down the action for the simplest $\mathcal{N}=1$ supersymmetric model. The Wess-Zumino model [15] consists of two complex scalar fields, ϕ and F, and a spinor field, ψ, and its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \overline{\phi})(\partial^\mu \phi) + i\psi\sigma^m u\partial_\mu \overline{\psi} + \overline{F}F, \tag{2.6}$$

which is invariant under the following (infinitesimal) supersymmetric transformations:

$$\begin{aligned}
\delta_\xi \phi &= \xi\psi, \\
\delta_\xi \psi &= F\xi - i\sigma^\mu \overline{\xi}\partial_\mu \phi, \\
\delta_\xi F &= -i\overline{\xi}\overline{\sigma}^\mu \partial_\mu \psi,
\end{aligned} \tag{2.7}$$

where ξ, $\overline{\xi}$ are anticommuting parameters of the transformation.

2.3 Superfields and Superspace

Although supersymmetric theories may be described adequately in terms of individual fields and their transformations as in the previous section, the elegance of the framework is somewhat hidden. This elegance is manifest, however, if the theory is restated in terms of two additional concepts: those of superfields and superspace to which we now turn.

Just as the translation group is parametrised by the spacetime coordinates, its extension, the super-translation group, may be parametrised by the coordinates of an extended space known as superspace [17]. This consists of ordinary spacetime coordinates augmented by the introduction of extra Grassmann coordinates (arranged in two-component spinors) denoted $\theta^\alpha, \overline{\theta}^{\dot{\alpha}}$. As Grassmann variables, these coordinates are anticommutative:

$$\begin{aligned}
\left\{\theta^\alpha, \theta^\beta\right\} &= 0, \\
\left\{\theta^\alpha, \overline{\theta}^{\dot{\beta}}\right\} &= 0, \\
\left\{\overline{\theta}^{\dot{\alpha}}, \overline{\theta}^{\dot{\beta}}\right\} &= 0,
\end{aligned} \tag{2.8}$$

and obey the following relations

$$\begin{aligned}
\int \mathrm{d}\theta^\alpha &= 0, \\
\int \mathrm{d}\theta_\alpha \, \theta^\beta &= \delta_\alpha^\beta, \\
\int \mathrm{d}^2\theta \, \theta\theta &= 1,
\end{aligned} \tag{2.9}$$

which are guaranteed by

$$\int \mathrm{d}\theta \equiv \frac{\partial}{\partial \theta}, \tag{2.10}$$

with similar relations for $\overline{\theta}_{\dot{\alpha}}$.

As the momentum operator acts as the generator of translations, a general transformation in the translation group may be written $e^{i(-x_\mu P^\mu)}$. Similarly, as $Q_\alpha, \overline{Q}_{\dot{\alpha}}$ are the SUSY generators, a general superspace translation may be written

$$e^{i\left(\theta Q+\overline{\theta Q}-x_\mu P^\mu\right)}, \tag{2.11}$$

where $x_\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}}$ are the superspace coordinates which parametrise the transformation.

By successively applying two SUSY transformations one finds, firstly, that such a transformation shifts the coordinates according to:

$$\begin{aligned} x^\mu &\mapsto x^\mu + i\theta\sigma^\mu\overline{\xi} - i\xi\sigma^\mu\overline{\theta}, \\ \theta &\mapsto \theta + \xi, \\ \overline{\theta} &\mapsto \overline{\theta} + \overline{\xi}, \end{aligned} \tag{2.12}$$

and, secondly, that the SUSY generators may be expressed as differential operators:

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu, \\ \overline{Q}_{\dot{\alpha}} &= \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \end{aligned} \tag{2.13}$$

Extending the concept of a field, which is a function of spacetime coordinates, to superspace, we get a superfield which is a function of both ordinary and Grassmann coordinates. A general superfield, $\mathcal{F}$ can be expressed in terms of ordinary fields (functions of spacetime coordinates only) by Taylor expanding in the Grassmann variables. Due to their anticommutativity, only terms up to quadratic in θ and $\overline{\theta}$ will be non-vanishing. As such, the general superfield may be written in terms of its component fields as

$$\begin{aligned} \mathcal{F}\left(x,\theta,\overline{\theta}\right) =&\phi(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) \\ &+ \theta\theta m(x) + \overline{\theta\theta}n(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) \\ &+ \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\theta\eta + \theta\theta\overline{\theta\theta}d(x). \end{aligned} \tag{2.14}$$

It should be noted that for a superfield, $\mathcal{F}$, the partial derivative, $\frac{\partial\mathcal{F}}{\partial\theta^\alpha}$, is not necessarily a superfield. What is needed is a covariant derivative, analogous to the covariant derivative of Riemannian spaces. Whilst the rigid superspace of global supersymmetry has no curvature, the torsion of superspace is non-zero, leading to a non-trivial covariant derivative. Just as left multiplication of group elements leads to the form of the SUSY generators, the form of the covariant derivative is revealed through right multiplication to be

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu, \\ \overline{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu, \end{aligned} \tag{2.15}$$

which obey the relations

$$\begin{aligned}\{D_\alpha, D_\beta\} &= \{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\alpha}\} = 0,\\ \{D_\alpha, \overline{D}_{\dot\alpha}\} &= -2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu,\end{aligned} \tag{2.16}$$

along with

$$\begin{aligned}\{D_\alpha, Q_\beta\} &= \{D_\alpha, \overline{Q}_{\dot\alpha}\} = 0,\\ \{\overline{D}_{\dot\alpha}, Q_\alpha\} &= \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0.\end{aligned} \tag{2.17}$$

As the covariant derivative of a superfield is itself a superfield with fewer components, it is clear that the general superfield is a reducible representation of the super-Poincaré group. In order to find irreducible representations, it is necessary to impose constraints on the general superfield, in such a way as to leave their x-dependence unrestricted, since we wish the component fields to behave as ordinary functions of spacetime coordinates. Two possible constraints lead to the concepts of the chiral and vector superfields.

2.3.1 Chiral and Anti-chiral Superfields

A possible constraint that one may impose on a general superfield, Φ, is that its covariant derivative vanishes:

$$\overline{D}_{\dot\alpha}\Phi = 0. \tag{2.18}$$

This reduces the general superfield to the following components:

$$\begin{aligned}\Phi(x,\theta,\overline{\theta}) =&\phi + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\partial_\mu\phi\theta\sigma^\mu\overline{\theta}\\ &+ \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta} + \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^2\phi.\end{aligned} \tag{2.19}$$

Other than the auxiliary field, this construction contains only a complex scalar field and a single (left-handed) Weyl spinor, and is known as a chiral superfield. The form of the chiral superfield is simplified by the introduction of "chiral coordinates", obeying $\overline{D}_{\dot\alpha}y^\mu = 0$ and given by

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}. \tag{2.20}$$

In terms of these coordinates, the chiral superfield is given by

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{2.21}$$

and we see that the $\overline{\theta}^{\dot\alpha}$'s do not feature.

Similarly, we may define an anti-chiral superfield, $\overline{\Phi}$, obeying

$$D_{\dot\alpha}\overline{\Phi} = 0, \tag{2.22}$$

and given in terms of the antichiral coordinates,

$$\begin{aligned}\overline{y}^\mu &= x^\mu - i\theta\sigma^\mu\overline{\theta}\\ &= y^\mu - 2i\theta\sigma^\mu\overline{\theta},\end{aligned} \tag{2.23}$$

by

$$\overline{\Phi}(y,\theta) = \overline{\phi}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) + \overline{\theta\theta F}(y). \tag{2.24}$$

Rewriting this in terms of the chiral coordinates, we have

$$\begin{aligned}\overline{\Phi} =&\overline{\phi}(y) + \sqrt{2}\,\overline{\theta\psi}(y) - 2i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\overline{\phi}(y) \\ &+ \overline{\theta\theta}\left(\overline{F}(y) + i\sqrt{2}\theta\sigma^{\mu}\partial_{\mu}\overline{\psi}(y) + \theta\theta\partial^{2}\phi(y)\right).\end{aligned} \tag{2.25}$$

At this point, we are able to appreciate the simplification of supersymmetric theories effected by the introduction of superfields, since the action of the Wess-Zumino model may now be written as [20]

$$S = \int \mathrm{d}^4\theta \mathrm{d}^4x\ \overline{\Phi}\Phi, \tag{2.26}$$

which reduces to (2.6) upon integration over $\theta, \overline{\theta}$ which may be seen using Eq. (2.9).

Similarly, it is equally straight-forward to introduce mass terms and interactions in the superfield formalism by means of the bilinear and trilinear terms of the "superpotential":

$$\mathcal{L}_P = \left[\int \mathrm{d}^2\theta \left(\frac{1}{2}m^2\Phi\Phi + \frac{1}{3}y\Phi\Phi\Phi\right)\right] + (\text{h.c.}), \tag{2.27}$$

where h.c. denotes the Hermitian conjugate.

2.3.2 Vector Superfields

An alternative constraint we may impose on a general superfield V is a reality condition:

$$V(x,\theta,\overline{\theta}) = \overline{V}(x,\theta,\overline{\theta}). \tag{2.28}$$

This leads to a superfield of the form

$$\begin{aligned}V(x,\theta,\overline{\theta}) =&C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \frac{i}{2}\theta\theta\left(M(x) + iN(x)\right) \\ &- \frac{i}{2}\overline{\theta\theta}\left(M(x) - iN(x)\right) + \theta\sigma^{\mu}\overline{\theta}A_{\mu}(x) \\ &+ i\theta\theta\overline{\theta}\overline{\lambda}(x) - i\overline{\theta\theta}\theta\lambda(x) \\ &+ \frac{1}{2}\theta\theta\overline{\theta\theta}D(x),\end{aligned} \tag{2.29}$$

where χ, λ are spinors, v_{μ} is a real vector, and C, M, N, D are real scalars.

In fact, this representation is still reducible and additional constraints must be imposed in order to make it irreducible.

However, as we wish the vector superfield to play the role of gauge field in the construction of supersymmetric Lagrangians, we will see that the additional degrees of freedom of the reducible representation may be removed by gauge-fixing. In order to see this, we redefine the superfield components according to

$$\begin{aligned}V(x,\theta,\overline{\theta}) =&C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \frac{i}{2}\theta\theta\left(M(x) + iN(x)\right) \\ &- \frac{i}{2}\overline{\theta\theta}\left(M(x) - iN(x)\right) + \theta\sigma^{\mu}\overline{\theta}A_{\mu}(x) \\ &+ i\theta\theta\overline{\theta}\left(\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^{\mu}\partial_{\mu}\chi(x)\right) - i\overline{\theta\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\overline{\chi}(x)\right) \\ &+ \frac{1}{2}\theta\theta\overline{\theta\theta}\left(D(x) - \frac{1}{2}\partial^2 C(x)\right).\end{aligned} \tag{2.30}$$

In this form, the component fields transform (in the Abelian case) in a consistent manner under the transformation

$$V \mapsto V + \Phi + \overline{\Phi} \tag{2.31}$$

(where Φ is chiral), according to

$$\begin{aligned}
C &\mapsto C + \phi + \overline{\phi}, \\
\chi &\mapsto \chi - i\sqrt{2}\psi, \\
M + iN &\mapsto M + iN - 2iF, \\
A_\mu &\mapsto A_\mu - i\partial_\mu \left(\phi - \overline{\phi}\right), \\
\lambda &\mapsto \lambda, \\
D &\mapsto D,
\end{aligned} \tag{2.32}$$

with a generalised transformation applying in the non-Abelian case as we shall see in the following section.

Note that the vector component transforms above as a gradient, in analogy with the corresponding field in non-supersymmetric gauge theories. Also note that the highest-dimensional components, λ, D, are invariant under this generalised gauge transformation. The remaining fields, however, are not invariant under the above transformation and, as such, if we are to consider this transformation as a gauge transformation then we must take these fields to be unphysical. Therefore, the additional constraint placed on the vector supermultiplet in order to ensure irreducibility should remove precisely these components.

The lower-dimensional components (being multiplied by fewer powers of θ, $\overline{\theta}$) may be removed by application of the covariant derivatives, rendering the following constructions gauge invariant:

$$\begin{aligned}
W_\alpha(x, \theta, \overline{\theta}) &= -\frac{1}{4}\overline{DD}e^{-2gV} D_\alpha e^{2gV}, \\
\overline{W}_{\dot{\alpha}}(x, \theta, \overline{\theta}) &= \frac{1}{4}DDe^{2gV}\overline{D}_{\dot{\alpha}}e^{-2gV},
\end{aligned} \tag{2.33}$$

which, in the Abelian case, simplify to

$$\begin{aligned}
W_\alpha(x, \theta, \overline{\theta}) &= -\frac{1}{4}\overline{DD}D_\alpha V, \\
\overline{W}_{\dot{\alpha}}(x, \theta, \overline{\theta}) &= \frac{1}{4}DD\overline{D}_{\dot{\alpha}}V.
\end{aligned} \tag{2.34}$$

Moreover, due to the nilpotency of the covariant derivatives, W_α, $\overline{W}_{\dot{\alpha}}$ are chiral and anti-chiral superfields respectively and are, therefore, irreducible. These fields contain only the components A_μ, λ^α and D. The A_μ field appears only in the form of the non-supersymmetric field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The D field is auxiliary and may be integrated out by means of its equation of motion, in a similar manner to the auxiliary field F of the chiral supermultiplet; the λ field is referred to as the gaugino. W_α, then, is the supersymmetric extension of the field strength.

As the majority of the components of V are gauge-dependent, we may remove them by fixing the gauge in such a way as to set all their values at zero. This simplifies our expression for V to

$$\begin{aligned}
V(x, \theta, \overline{\theta}) = &- \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\theta\lambda(x) \\
&+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D(x).
\end{aligned} \tag{2.35}$$

This is known as the Wess-Zumino gauge and is in fact only a partial gauge-fixing condition. We are left with a residual freedom to choose $\phi - \overline{\phi}$, the gradient of which defines the gauge transformation of the vector field; that is, we still retain the original (spacetime) gauge-symmetry of the non-supersymmetric theory.

It has been shown that chiral and vector superfields are sufficient for the construction of *all* renormalisable supersymmetric Lagrangians [18].

2.4 Supergauge Theory

As the vector superfield transforms in an appropriate manner for the gauge field, we now wish to introduce matter fields which undergo a gauge transformation in an equally familiar fashion. This is achieved by means of the chiral superfield. Under the transformation

$$\begin{aligned}\Phi &\mapsto e^{-i\Lambda^a T_a}\Phi,\\ \overline{\Phi} &\mapsto \overline{\Phi}e^{i\overline{\Lambda}^a T_a}\end{aligned} \tag{2.36}$$

(where the T_a are the symmetry group generators), a chiral superfield maps into a second chiral superfield only if Λ itself is also a chiral multiplet. Whilst the kinetic term for a chiral superfield is invariant under global transformations (with constant Λ), under local (gauge) transformations, it is found to transform according to

$$\overline{\Phi}\Phi \mapsto \overline{\Phi}e^{i\left(\overline{\Lambda}-\Lambda\right)}\Phi. \tag{2.37}$$

As $i\left(\overline{\Lambda}-\Lambda\right)$ is Hermitian, invariance may be restored by introducing a vector superfield

$$\mathcal{L}_{\text{kin}} = \int \mathrm{d}^4\theta\; \overline{\Phi}e^V\Phi, \tag{2.38}$$

transforming as

$$\begin{aligned}e^V &\mapsto e^{-i\overline{\Lambda}}e^V e^{i\Lambda},\\ &= e^{-i\overline{\Lambda}}\left(1+V+\frac{1}{2}V^2+\dots\right)e^{i\Lambda},\end{aligned} \tag{2.39}$$

(which reduces to the transformation given for vector superfields in the previous section (Eq. 2.32) in the case of infinitesimal Abelian gauge transformations).

In fact then, the general gauge transformation for a vector superfield is given by (2.39). Calculations involving this term are simplified by working in the Wess-Zumino gauge as, in this gauge, we have

$$\begin{aligned}V^2 &= -\frac{1}{2}\theta\theta\overline{\theta\theta}A_\mu A^\mu,\\ V^n &= 0 \qquad n \geq 3.\end{aligned} \tag{2.40}$$

In components, the kinetic terms of the Lagrangian are given by

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& \overline{F}^a F^a - i\overline{\psi}^a\overline{\sigma}^\mu(D_\mu\psi)^a - \overline{(D^\mu\phi)}^a(D_\mu\phi)^a\\ &+ if^{abc}\overline{\phi}^a D^b\phi^c - f^{abc}\sqrt{2}\left(\overline{\phi}^a\lambda^b\psi^c - \overline{\psi}^a\overline{\lambda}^b\phi^c\right),\end{aligned} \tag{2.41}$$

where

$$D_\mu\phi^a \equiv \partial_\mu\phi^a - \frac{1}{g}f^{abc}A_\mu^b\phi^c \tag{2.42}$$

as in the non-supersymmetric case.

All that remains is to introduce a kinetic term for the gauge fields analogous to the $F_{\mu\nu}F^{\mu\nu}$ term in the non-supersymmetric case. As we don't wish to introduce the unphysical components of the vector superfield, this is achieved via the multiplet's irreducible counterparts W_α, $\overline{W}_{\dot{\alpha}}$. A term of the form

$$\mathcal{L}_{\text{gauge}} = \left[\int \mathrm{d}^2\theta \, \frac{1}{4}\mathrm{tr}\left(W^\alpha W_\alpha\right)\right] + (\text{h.c.}) \tag{2.43}$$

is gauge invariant and irreducible and, in the component formalism, reduces to

$$\mathcal{L}_{\text{gauge}} = -i\lambda\sigma^\mu\partial_\mu\overline{\lambda} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{16}F^{\xi\rho}F^{\sigma\tau}\epsilon_{\xi\rho\sigma\tau} + \frac{1}{2}D^2, \tag{2.44}$$

where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu. \tag{2.45}$$

We see from this that the vector and chiral spinor components behave just as we would expect for a gauge theory in the non-supersymmetric case. Thus, the successful gauge theories of particle physics may be extended to the supersymmetric arena with the introduction of scalar and spinor partners to the fermion and vector fields respectively.

2.5 Motivation for Supersymmetry

Having introduced the concept of supersymmetry, we now consider the motivation for its introduction. Reasons for studying supersymmetry are varied and we consider four in the following sections. In addition to these, there is also a compelling feature of supersymmetric theories that arises when their renormalisation is studied but discussion of this will be deferred until chapter 4.

2.5.1 Dark Matter

As mentioned in §2.2, the graded structure of the supersymmetry algebra allows for a non-trivial interaction between the internal symmetries and the symmetries of the super-Poincaré group. The subset of internal symmetry generators which have non-zero commutators with the SUSY generators are known as the R-symmetry generators. The form of the R-symmetry is constrained, however, and in the case of $\mathcal{N} = 1$ SUSY may only take the form of a $U(1)$ group; that is, there may be at most one generator which does not commute with $Q, \overline{Q}$.

In the minimal supersymmetric standard model (MSSM), the $U(1)$ R-symmetry is broken to a Z_2 "R-parity" which is taken to be conserved [20]. Interactions in the MSSM are restricted by R-partiy in such a way that superpartners may only be produced or annihilated in pairs. R-parity was introduced to the MSSM as a means of ensuring the stability of the proton but it also provides a possible explanation for the Cold Dark Matter of the Λ_{CDM} model of cosmology [21]. Modern cosmology suggests that only 20% of the universe is composed of "baryonic" (standard model) matter, whilst the remaining 80% consists of dark matter [22] which shows itself only through its gravitational influence. Under R-parity, the lightest superpartner would be stable and could, therefore, account for at least some of this dark matter.

2.5.2 The Running of the Couplings

The couplings of the three forces of the Standard Model vary with energy scale due to the scale-dependence of the renormalisation procedure. This is known as the running of the couplings. The couplings appear almost to meet at an energy of around 10^{16}GeV, suggesting a unification of the symmetries at this scale into a larger symmetry group. However, as the standard model couplings do not quite meet at a single energy scale, such "Grand Unified Theories" require the introduction of non-Standard Model physics at an intermediate scale. One candidate for this new physics is supersymmetry with the masses of the superpartners being of the order of 1000 GeV [23].

2.5.3 The Hierarchy Problem

A physically relevant momentum cutoff in the form of a "matching scale", between the low-energy effective theory and the UV theory, leads to a problem of naturalness in the Standard Model. UV-sensitive parameters with a value of the order of the cutoff are said to be natural; if the observed value of such a parameter is much less than the cutoff, then a high degree of fine-tuning at the cutoff scale is necessary to produce the observed value and the theory is said to be un-natural. There are a number of possible solutions to this problem, one of which is simply to assume that the UV theory (of which we have little knowledge) will provide an explanation for the fine-tuning. Another possibility is the introduction of new physics at a scale around the value of the UV-sensitive parameter, thus lowering the matching scale to the (now natural) value of the parameter.

A specific example of this problem is given by the Higgs mass. The masses of all particles in the standard model are generated by the non-zero vacuum expectation value of the Higgs field. As this requires the Higgs field to be coupled to all massive particles, the mass of the Higgs boson is driven up by quadratic divergences coming from loop diagrams involving any massive particle. This is problematic since, although the Higgs boson has not yet been observed in experiment, its mass is constrained by electroweak theory to be relatively low. Hence, any contributions to the Higgs mass from loop diagrams must be cancelled off, which leads to fine-tuning and un-naturalness. This is known as the hierarchy problem. A (partial) solution to this problem is also provided by supersymmetry [24], since the quadratic divergences coming from bosonic loop contributions to the Higgs mass are cancelled in supersymmetric theories by the corresponding fermionic superpartner loops. This is only a partial solution as the logarithmic divergences remain to be dealt with, and thus seem to require a lesser, but still non-zero, degree of fine-tuning. This is known as the "Little Hierarchy Problem".

2.5.4 String Theory

The two most successful theories of modern physics, namely Quantum Mechanics and General Relativity, are mutually inconsistent, implying that the "Theory of Everything" has yet to be found and must reduce to the correct theory in the appropriate limit. A consistent theory of quantum gravity has proved elusive but of the few current contenders, arguably the most successful (and certainly the most widely studied) is String Theory. As a string travels through space, it describes a two-dimensional world-sheet in spacetime. As each point on this world-sheet is located at a unique point in spacetime, the spacetime coordinates take values on the worldsheet and may be viewed as bosonic degrees of freedom in a two-dimensional space. This only allows for bosonic fields, however, and in order for fermionic fields to feature (which of course they must) there must be fermionic degrees of freedom on the worldsheet. These correspond to the Grassmann coordinates of superspace. Therefore, in order for String Theory to describe fermions, it is necessary to introduce supersymmetry [25].

Chapter 3

$\mathcal{N} = \frac{1}{2}$ Supersymmetry

3.1 Introduction to Non-anticommutativity

With the introduction of a new set of coordinates for supersymmetry, it is natural to ask if any more structure can be given to spacetime. This idea has been studied in the past in the fields of non-commutative geometry and non-commutative field theory [26, 27], in which spacetime coordinates are taken to obey a commutation relation of the form

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}. \tag{3.1}$$

The original motivation for the introduction of such a relation was to avoid the divergences arising in interacting field theories, and it has also been noted that an uncertainty in position of the above form could prevent the formation of singularities in General Relativity [28].

Combining the ideas of supersymmetry and non-commutative geometry, we can also ask what would be the consequences of a similar deformation of the superspace coordinates [29–35]. That is: rather than a fully anticommutative superspace, we introduce a deformation to θ^α in the form of a Clifford algebra

$$\begin{aligned}
\left\{\theta^\alpha, \theta^\beta\right\} &= C^{\alpha\beta}, \\
\left\{\theta^\alpha, \overline{\theta}^{\dot{\beta}}\right\} &= 0, \\
\left\{\overline{\theta}^{\dot{\alpha}}, \overline{\theta}^{\dot{\beta}}\right\} &= 0,
\end{aligned} \tag{3.2}$$

where $C^{\alpha\beta}$ is symmetric. Note that the deformation only affects θ^α and not $\overline{\theta}_{\dot{\alpha}}$.

Such non-anticommutativity was first put forward in relation to String Theory [36] as a possible outcome of introducing supersymmetry on the worldsheet. It was later shown [37, 38] that such a deformation arises naturally as a consequence of introducing a constant graviphoton background in Heterotic and Type II String theories. Our approach, however, is to take the deformation in Eq. (3.2) as axiomatic and investigate the properties — in particular the renormalisation — of field theories formulated in this deformed superspace. Whilst non-anticommutative field theories are of limited application (though more will be said on this in §3.3), they provide some interesting examples of field theories that turn out to be renormalisable on careful inspection, despite containing terms of mass-dimension higher than four. This is a point that we will return to shortly.

3.2 Properties of Non-anticommutativity

In terms of the spacetime coordinates x^μ, the expressions for the covariant derivatives contain both θ's and $\overline{\theta}$'s. In terms of the chiral coordinates given in (2.20), however, they take the form

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \\ \overline{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}, \end{aligned} \tag{3.3}$$

and since we wish to be able to continue to define chiral and antichiral superfields in terms of these derivatives in the non-anticommutative case, we take these expressions to define the derivatives [37]. Therefore, we must work in chiral coordinates y^μ, which we take to commute with themselves and with θ, $\overline{\theta}$. This leads directly to the commutation relations

$$\begin{aligned} [x^\mu, \theta^\alpha] &= iC^{\alpha\beta}\sigma^\mu_{\beta\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}, \\ [x^\mu, x^\nu] &= \overline{\theta\theta}C^{\mu\nu}, \end{aligned} \tag{3.4}$$

where

$$C^{\mu\nu} \equiv C^{\alpha\beta}\epsilon_{\beta\gamma}\left(\sigma^{\mu\nu}\right)_\alpha{}^\gamma \tag{3.5}$$

and

$$\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta \equiv \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\zeta}}\left(\overline{\sigma}^\nu\right)^{\dot{\zeta}\beta} - \sigma^\nu_{\alpha\dot{\zeta}}\left(\overline{\sigma}^\nu\right)^{\dot{\zeta}\beta}\right), \tag{3.6}$$

so we see that the space is non-commutative as well as non-anticommutative.

Furthermore, since θ, $\overline{\theta}$ now obey different anticommutation relations, they cannot be complex conjugates and so cannot be defined in Minkowski space. For this reason, we find it necessary to work in Euclidean space.

Since we are working in chiral coordinates, we must also express the SUSY operators Q, $\overline{Q}$ in these terms

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha}, \\ \overline{Q}_{\dot{\alpha}} &= -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}. \end{aligned} \tag{3.7}$$

With expressions for the covariant derivatives and supercharges, we may calculate the remaining commutation relations for the SUSY algebra. The supercharges obey

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0, \\ \left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} &= -2\sigma^\mu_{\alpha\dot{\beta}}\frac{\partial}{\partial y^\mu}, \\ \left\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} &= -4C^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\frac{\partial^2}{\partial y^\mu \partial y^\nu}, \end{aligned} \tag{3.8}$$

whilst the derivatives obey

$$\begin{aligned} \{D_\alpha, D_\beta\} &= 0, \\ \left\{\overline{D}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\right\} &= 0, \\ \left\{D_\alpha, \overline{D}_{\dot{\beta}}\right\} &= 2i\sigma^\mu_{\alpha\dot{\beta}}\frac{\partial}{\partial y^\mu}, \end{aligned} \tag{3.9}$$

and we have

$$\{D_\alpha, Q_\beta\} = \left\{D_\alpha, \overline{Q}_{\dot\beta}\right\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} = \left\{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0. \tag{3.10}$$

These relations are identical to those of the undeformed case, the only exception being the anticommutator of two $\overline{Q}$'s in (3.8).

As the covariant derivatives were unaltered by the superspace deformation, the form of the chiral and antichiral superfields also remains unchanged:

$$\begin{aligned}\Phi(y) =& \phi(y) + \theta\psi(y) + \theta\theta F(y),\\ \overline{\Phi}(y) =& \overline{\phi}(y) + \sqrt{2}\theta\overline{\psi}(y) - 2i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{\phi}(y)\\ &+ \overline{\theta\theta}\left(\overline{F}(y) + i\sqrt{2}\theta\sigma^\mu\partial_\mu\overline{\psi}(y) + \theta\theta\partial^2\phi(y)\right).\end{aligned} \tag{3.11}$$

3.3 Physicality of Non-anticommutative Models

As was noted in §3.1, the first $\mathcal{N} = \frac{1}{2}$ theories arose as a particular low-energy limit of string theory. It was already known that D3-branes living in certain (NSNS) backgrounds give rise to a low-energy theory with non-commutative space-time geometry. Non-anticommutativity was then found in a similar fashion as the low-energy limit of a D3-brane living in a Ramond-Ramond self-dual graviphoton background [38]. The $\mathcal{N} = \frac{1}{2}$ theories thus derived were constrained to live in Euclidean space in order for the chiral and anti-chiral coordinates to have differing anticommutativity properties. This is consistent with the necessarily Euclidean space in which the graviphoton field may be taken to be self-dual. Given the Euclidean and non-Hermitian nature of $\mathcal{N} = \frac{1}{2}$ theories, it is difficult to see how a phenomenological interpretation may be applied to such models. However, there are some subtleties regarding this matter.

Since non-anticommutative theories are (generally) renormalisable, it seems plausible that a Wick rotation to Minkowski space could have phenomenological implications [39]. However, a simple Wick rotation is taken to be prohibited in Ref. [40] since it leads to violation of the Jacobi identites of the deformation. On the other hand, since the Jacobi identity is not required to hold in non-associative algeras, dropping the assumption of associativity can lead to a consistent Minkowski formulation of non-anticommutativity [33, 41]. Since the form of deformation given in Eq. (3.2) is particular to Euclidean space, the Minkowski formulation has constraints placed on it relating the form of the deformation in the chiral and anti-chiral cases.

Turning now to the subject of Hermiticity, though the Euclidean theories are non-Hermitian, there is evidence that this need not render them unphysical. The subject of pseudo- or crypto-Hermiticity is raised in Refs. [42–44], where it is shown that non-Hermitian Hamiltonians may, in some cases, still lead to a Hilbert space of state vectors that is Hermitian and unitary, and which has a real spectrum. This is achieved through an equivalence relation between the Hamiltonian in question and an explicitly Hermitian Hamiltonian, and has been shown, in Ref. [45], to be the case for non-anticommutative theories.

In any case, though the non-Hermitian, Euclidean theories discussed in the main chapters of this work do not necessarily have immediate physical interpretations, they nevertheless provide an insight into the dynamics of non-anticommutativity and the low-energy effects of string backgrounds.

3.4 Star Products

The simplest way to implement non-anticommutativity is in terms of the Moyal Star-product defined by

$$
\begin{aligned}
f(\theta) \star g(\theta) &= f(\theta) exp\left(-\frac{1}{2}C^{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\theta}}{\partial\theta^\beta}\right) g(\theta) \\
&= f(\theta)\left(1 - \frac{1}{2}C^{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta} - \det C \frac{\overleftarrow{\partial}}{\partial\theta\theta}\frac{\overrightarrow{\partial}}{\partial\theta\theta}\right) g(\theta).
\end{aligned}
\tag{3.12}
$$

With products of θ's in an undeformed theory replaced by the star-product, the C-dependence has already been incorporated into the model and so the θ's that appear in star products may then be treated as Grassmannian.

The star-products of θ's can be calculated as follows

$$
\begin{aligned}
\theta^\alpha \star \theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta + \frac{1}{2}C^{\alpha\beta}, \\
\theta^\alpha \star \theta\theta &= C^{\alpha\beta}\theta_\beta, \\
\theta\theta \star \theta^\alpha &= -C^{\alpha\beta}\theta_\beta, \\
\theta\theta \star \theta\theta &= -\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}C^{\alpha\gamma}C^{\beta\delta} \\
&= -\det C.
\end{aligned}
\tag{3.13}
$$

As θ appears explicitly in $\overline{Q}$, star-products will not be invariant under $\overline{Q}$. So, whilst the Q's remain a symmetry of a non-anticommutative theory, the invariance under $\overline{Q}$ is lost, effectively breaking half of the supersymmetry: hence the name $\mathcal{N} = \frac{1}{2}$. One consequence of this is that the chiral and antichiral fields will transform differently under $\mathcal{N} = \frac{1}{2}$ transformations. Thus the theory is not Hermitian; however, whilst this would seem to be un-natural, we will find in the next chapter that it is this non-Hermiticity that allows for the renormalisability of non-anticommutative theories.

The star-product of two chiral superfields is given by

$$
\begin{aligned}
\Phi_1(y,\theta) \star \Phi_2(y,\theta) = \Phi_1(y,\theta)\Phi_2(y,\theta) &- C^{\alpha\beta}\psi_{1\alpha}(y)\psi_{2\beta}(y) \\
&+ \sqrt{2}C^{\alpha\beta}\theta_\beta\left(\psi_{1\alpha}(y)F_2(y) - \psi_{2\alpha}(y)F_1(y)\right) \\
&- \det C F_1(y)F_2(y),
\end{aligned}
\tag{3.14}
$$

which is itself a chiral superfield, as in the undeformed case. Note that the extra terms contain only one factor of θ, implying that the mass term of the Lagrangian will acquire no deformation. Similarly, the star-product of two antichiral superfields is itself an antichiral superfield.

3.5 Construction of $\mathcal{N} = \frac{1}{2}$-Invariant Lagrangians

In much the same way as in the undeformed case, we may now construct deformed Lagrangians, with star-products replacing ordinary products of superfields. We take as an example the Wess-Zumino model with a superpotential. As mentioned above, the holomorphic mass term provides no new terms in the Lagrangian. Simlarly, as each factor of C from the star-product is accompanied by a decrease in the power of

θ, it should be clear that the antiholomophic part of the superpotential also leaves the action unchanged. With the only additional term from the deformation arising in the star-product of three chiral superfields, the Lagrangian is given by [37]

$$\begin{aligned}\mathcal{L} &= \int \mathrm{d}^4\theta\, \overline{\Phi}\Phi + \left[\int \mathrm{d}^2\theta \left(\frac{1}{2}m\Phi \star \Phi + \frac{1}{3}y\Phi \star \Phi \star \Phi\right) + \text{h.c.}\right] \\ &= \mathcal{L}_{\text{undeformed}} + \frac{1}{6}y\det(C)F^3.\end{aligned} \tag{3.15}$$

This model will be the subject of chapter 6 where we will derive this action by an alternative method and investigate its renormalisation.

Gauge theories may also be extended to the deformed case, with the vector superfield (in Wess-Zumino gauge) being modified according to

$$\begin{aligned}V =& -\theta\sigma^\mu\overline{\theta}A_\mu(y) + i\theta\theta\overline{\theta}\overline{\lambda}(y) - i\overline{\theta}\overline{\theta}\theta\left(\lambda(y) + \frac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot{\gamma}}\left\{\overline{\lambda}^{\dot{\gamma}}, A_\mu\right\}\right) \\ &+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D(y) - i\partial_\mu A^\mu\right),\end{aligned} \tag{3.16}$$

where we have also re-expressed the superfield in terms of the chiral coordinates.

Defining the field strengths for V as in the undeformed case (Eq. (2.33)), but with ordinary products replaced by star products, we find that the kinetic term for vector superfields is given by

$$\mathcal{L}_{\text{gauge}} = -i\lambda\sigma^\mu\partial_\mu\overline{\lambda} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2 - igC^{\mu\nu}\mathrm{tr}(F_{\mu\nu}\overline{\lambda}\overline{\lambda}) + g^2|C|^2\mathrm{tr}(\overline{\lambda}\overline{\lambda})^2, \tag{3.17}$$

so the additional terms coming from the deformation are merely a $\overline{\lambda}\overline{\lambda}A_\mu$ term and four-point gaugino interaction.

Finally, we must consider the deformation of the kinetic terms for chiral and antichiral superfields with a gauge symmetry. As in the undeformed case, powers of V higher than two still vanish (in the Wess-Zumino gauge) and we have

$$V \star V = \overline{\theta}\overline{\theta}\left(-\frac{1}{2}\theta\theta A_\mu A^\mu - \frac{1}{2}C^{\mu\nu}A_\mu A_\nu + \frac{i}{2}\theta_\alpha C^{\alpha\beta}\sigma^\mu_{\beta\dot{\alpha}}\left[A_\mu, \overline{\lambda}^{\dot{\alpha}}\right] - \frac{1}{8}|C|^2\overline{\lambda}\overline{\lambda}\right), \tag{3.18}$$

giving [46]

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \overline{\Phi} \star e^{2gV} \star \Phi\big|_{\theta\theta\overline{\theta}\overline{\theta}} \\ &= \overline{\Phi}\left(1 + 2gV + 2g^2V \star V\right)\Phi\big|_{\theta\theta\overline{\theta}\overline{\theta}} \\ &= \overline{F}F - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi - \overline{D^\mu\phi}D_\mu\phi + \overline{\phi}D\phi + \frac{i}{\sqrt{2}}\left(\overline{\phi}\lambda\psi - \overline{\psi}\overline{\lambda}\phi\right) \\ &\quad + igC^{\mu\nu}\overline{\phi}F_{\mu\nu}F - \frac{1}{4}g^2|C|^2\overline{\phi}\overline{\lambda}\overline{\lambda}F + \sqrt{2}gC^{\mu\nu}D_\mu\overline{\phi}\overline{\lambda}\overline{\sigma}_\nu\psi.\end{aligned} \tag{3.19}$$

In order to construct a Lagrangian for an $\mathcal{N} = \frac{1}{2}$ gauge theory with a chiral multiplet, we must choose a representation of the gauge symmetry under which the multiplet will transform. The most obvious possibility is for the chiral fields to transform under the fundamental and anti-fundamental representations, in which case the chiral fields multiply the generators of the symmetry group in the usual way (as is found, for example, in the standard model). An alternative, however, which will be required in chapter 8, is for the chiral fields to transform according to the adjoint representation: that is, in the same representation as the gauge fields. The form of the Lagrangian for such a theory with a superpotential is given by [47]

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{g^2}\left(-\frac{1}{4}F^{\mu\nu a}F^b_{\mu\nu} - i\overline{\lambda}^a\overline{\sigma}^\mu\left(D_\mu\lambda\right)^b + \frac{1}{2}D^aD^b\right)\delta^{ab} \\
& - \frac{i}{2g^2}C^{\mu\nu}d^{abc}F^a_{\mu\nu}\overline{\lambda}^b\overline{\lambda}^c \\
& + \overline{F}F - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi - \overline{D^\mu\phi}D_\mu\phi + if^{abc}\overline{\phi}^aD^b\phi^c - f^{abc}\sqrt{2}\left(\overline{\phi}^a\lambda^b\psi^c - \overline{\psi}^a\overline{\lambda}^b\phi^c\right) \\
& + C^{\mu\nu}d^{abc}\left(\sqrt{2}\overline{D_\mu\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^c + i\overline{\phi}^aF^b_{\mu\nu}F^c\right) \\
& + \frac{1}{2}\Big(yd^{abc}\phi^a\phi^bF^c - yd^{abc}\phi^a\psi^b\psi^c \\
& + \overline{y}d^{abc}\overline{\phi}^a\overline{\phi}^b\overline{F}^c - \overline{y}d^{abc}\overline{\phi}^a\overline{\psi}^b\overline{\psi}^c\Big) \\
& + \frac{1}{3}i\overline{y}C^{\mu\nu}f^{abc}D_\mu\overline{\phi}^aD_\nu\overline{\phi}^b\overline{\phi}^c - \frac{1}{3}i\overline{y}C^{\mu\nu}d^{abc}d^{cde}F^d_{\mu\nu}\overline{\phi}^a\overline{\phi}^b\overline{\phi}^c,
\end{aligned}
\tag{3.20}
$$

which is invariant under the infinitesimal transformations

$$
\begin{aligned}
\delta_\xi A^a_\mu &= -i\overline{\lambda}^a\overline{\sigma}_\mu\xi, \\
\delta_\xi\lambda^a_\alpha &= i\xi_\alpha D^a + \left(\sigma^{\mu\nu}\right)_\alpha\left[F^a_{\mu\nu} + \frac{1}{2}iC_{\mu\nu}d^{abc}\overline{\lambda}^b\overline{\lambda}^c\right], \\
\delta_\xi\overline{\lambda}^a_{\dot\alpha} &= 0, \\
\delta_\xi D^a &= -\xi\sigma^\mu D_\mu\overline{\lambda}^a, \\
\delta_\xi\phi^a &= \sqrt{2}\xi\psi^a, \\
\delta_\xi\overline{\phi}^a &= 0, \\
\delta_\xi\psi^\alpha &= \sqrt{2}\xi^\alpha F^a, \\
\delta_\xi\overline{\psi}^a_{\dot\alpha} &= -i\sqrt{2}\left(D_\mu\overline{\phi}\right)^a\left(\xi\sigma^\mu\right)_{\dot\alpha}, \\
\delta_\xi F^a &= 0, \\
\delta_\xi\overline{F}^a &= -i\sqrt{2}(D_\mu\overline{\psi})^A\overline{\sigma}^\mu\xi + 2f^{bca}\left(\overline{\phi}^b\xi\lambda^c\right) + 2C^{\mu\nu}d^{abc}D_\mu\left(\overline{\phi}^b\xi\sigma_\nu\overline{\lambda}^c\right),
\end{aligned}
\tag{3.21}
$$

where the f^{abc} and d^{abc} are the antisymmetric and symmetric structure constants for the gauge group. Note the different transformations for chiral and antichiral fields.

Various $\mathcal{N} = \frac{1}{2}$ field theories have been studied in both the superfield and component formalisms [29–36, 38, 48, 49]. Whilst the superfield formalism provides an understanding of the origins of a given theory, it also hides the phenomenological aspects of the theory somewhat. For this reason, we choose to work in the uneliminated component formalism (without integrating out auxiliary fields) and, for completeness, we repeat some calculations in the eliminated formalism as a consistency check. Pure non-Abelian gauge theory and theories with chiral matter in both the fundamental and adjoint representations of $SU(N) \otimes U(1)$ have been studied in Refs. [50–53] and an attempt to include a superpotential into such a model in Ref. [54]. The latter model met with difficulties that will be discussed in chapter 8. In chapter 6, we investigate the renormalisation of a pure chiral model without gauge interactions, expanding this to an Abelian gauge theory in chapter 7. We begin with a rather different model, however, which we will turn to in chapter 5: namely, the two-dimensional non-linear sigma-model with $\mathcal{N} = 2$ supersymmetry.

Chapter 4

Renormalisation

The concept of renormalisation is discussed in this chapter, firstly in a broad sense and then, in §4.2, focusing specifically on the renormalisation of supersymmetric theories. Finally, in §4.3, the renormalisation of $\mathcal{N} = \frac{1}{2}$ theories is discussed in preparation for the remaining chapters.

4.1 Renormalisation Basics

As the amplitude for a process in an interacting field theory includes the sum of all possible ways that such a process could occur, including all values of momentum carried by intermediate particles, the calculation of such an amplitude involves integrating over all possible internal momenta. As these momenta are unconstrained, the integral must be evaluated over all real values, in some instances causing the result to diverge. The divergences arising in momentum integrals are of two types, referred to as ultraviolet and infrared. Ultraviolet divergences arise from the range of the integral in which the momentum takes indefinitely large values. An integrand in which the denominator contains a higher power of the momentum k than the numerator will tend to zero as $|k| \to \infty$, rendering the high-momentum region of the integral finite. Hence, ultraviolet divergences only arise when the numerator has a power greater than or equal to that of the denominator. Infrared divergences, on the other hand, occur when the power of the numerator is insufficient to cancel the denominator as the denominator tends to zero: that is, as $|k| \to 0$ in the massless case or $|k| \to m$ in the massive case. Clearly, an infinite probability amplitude is nonsensical and steps must be taken to obtain a sensible answer to amplitude calculations. If a model has only a finite number of one-particle irreducible divergent amplitudes (even if there are contributions to those amplitudes at all orders), the parameters of the model may be adjusted or renormalised in order to remove those divergences; in this case, the theory is said to be renormalisable. Otherwise, with infinitely many divergent amplitudes, infinitely many parameters would require renormalisation and the theory is said to be non-renormalisable. The parameters which will require renormalisation are those corresponding to the divergent amplitudes; that is, if the two-point function is a divergent amplitude, the mass will require renormalisation. Similarly, if there is a divergent amplitude, for example, of the form $\psi\psi\phi$ then the coupling for this interaction as present in the action will require renormalisation. It should be noted, however, that the contributions to a divergent amplitude do not depend solely on the corresponding interaction; for instance, a pair of three-point interactions may give a contribution to the two-point function and cause a renormalisation of the mass. Generally, the renormalisability or otherwise of a model is dictated by the mass-dimension of its couplings: if the lowest-dimension coupling is dimensionless, the model will be renormalisable, if negative it will be non-renormalisable [1]. If the mass-dimension is positive, there will only be a finite number of contributions to the divergent amplitudes and, in this case, the model is said to be super-renormalisable. This general rule is found not to apply, however, to $\mathcal{N} = \frac{1}{2}$ theories as will be discussed in §4.3. Each of the two types of divergence has associated with it methods by which to remove them. These methods are discussed below. Although the Standard Model is fully renormalisable, the fact that the divergences appear at all strongly suggests that the Standard Model (and indeed quantum field theory in general) is only an effective theory, valid up to a certain energy scale. Above this scale, a more fundamental theory is required which will reduce to the Standard Model in the low-energy limit. However, any underlying theory that appears to take the form of a quantum field theory at low energies must give sensible results, so renormalisability is still a strong indicator of realistic theories. Indeed, it may be shown that apparently unrenormalisable terms of the action for a fundamental theory defined at high energy will become irrelevant at much lower energies. So the effective field theory defined at low energy *must* contain only the renormalisable terms of the underlying action.

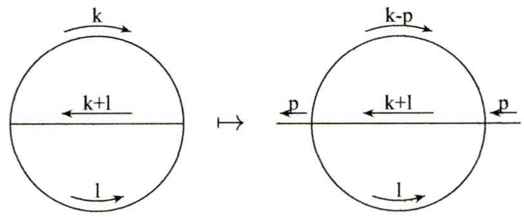
4.1.1 Ultraviolet Divergences

The removal of ultraviolet divergences takes place in two stages. Firstly, an extra component is introduced to the momentum integral in such a way as to render the integral finite but to reduce to the original (divergent) result in the appropriate limit. This component is known as a regulator and its insertion as regularisation. This allows the integral to be computed in an unambiguous way and the result parametrised in terms of the regulator. There are various methods of regularisation, the most obvious of which is to introduce an upper bound for the domain of integration known as the momentum cut-off. Whilst this method has the advantage of being intuitive, it is otherwise rather unsatisfactory as it breaks Lorentz symmetry. The approach used in our calculations is that of dimensional regularisation. Whilst the motivation behind this method is less immediately obvious, it has the advantages of being Lorentz and gauge invariant [55]. This method has its basis in the fact that the integrals under consideration are not necessarily divergent in a spacetime of arbitrary dimension. As a lower-dimensional spacetime requires integration over fewer momentum components, the integrals will generally converge. Thus, we evaluate the convergent integral in a d-dimensional space and continue the result analytically back to the 4-dimensional case (assuming a 4-dimensional theory). In order for the action to remain a scalar functional in the d-dimensional space, a mass parameter μ is introduced which must of course vanish for $d = 4$. Thus we are led to an action of the form

$$S = \mu^{\epsilon} \int \mathrm{d}^d x \; \mathcal{L}_{d=4}, \tag{4.1}$$

where $\epsilon = 4 - d$.

The result of a loop calculation is dependent on the external momenta; if no such momenta are present, a fictitious momentum is fed through at least one of the graph's propagators and may be taken to zero after the calculation. Thus a graph of the form



(4.2)

would be given by the integral

$$\int \frac{\mathrm{d}^d k \ \mathrm{d}^d l}{(k-p)^2 l^2 (k+l)^2}. \tag{4.3}$$

Useful formulae for evaluating integrals of this type may be found in appendix A.

Integrals with equal powers of momentum in the numerator and denominator will depend on the logarithm of the regulator and hence are known as logarithmically divergent integrals. For each extra power of k in the numerator, the degree of divergence increases; a graph of the form (4.2) is quadratically divergent and so forth.

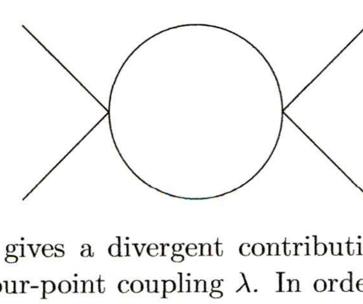
The second step in the removal of divergences is to renormalise the parameters of the theory. By taking the regulator to the appropriate limit *after* evaluation of the integral, we can obtain an expression for the divergence. Since the bare parameters of the action (masses, couplings and field strengths) are unobservable, we may now rescale them in order to absorb the divergences, which leads to an effective action containing the observable parameters. More precisely, we relate the bare and observable parameters by

$$\phi_B = Z_\phi^{\frac{1}{2}} \phi, \qquad m_B = Z_m m, \qquad g_B = Z_g g, \tag{4.4}$$

with similar relations for the remaining fields and couplings. The Z's are divergent scaling factors known as the renormalisation constants. These are chosen such as to cancel the divergent contributions of loop calculations.

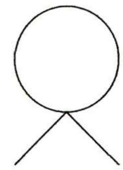
There is a certain ambiguity present in this procedure in that there is no fixed way to choose the Z's. The specific prescription for this choice is given in a set of renormalisation conditions which can take numerous forms, depending on how much of the loop-integral expression we wish to remove from the action. The result of calculation of a physical process is necessarily independent of the choice of renormalisation scheme. As we are investigating the renormalisation procedure itself for $\mathcal{N} = \frac{1}{2}$ theories, we are uninterested in any finite terms associated with the momentum integrals; as such, we choose to work in the "minimal subtraction" scheme in which we keep only the pole terms and neglect any finite parts of calculations.

The renormalisation constants are calculated order by order in the coupling, each order appearing in the calculation of the next. To demonstrate this, a one-loop diagram of the form



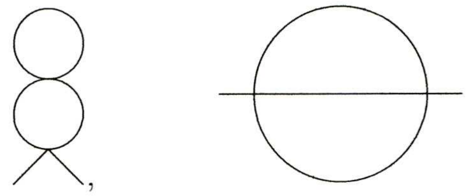
(4.5)

in a ϕ^4 scalar field theory gives a divergent contribution to the four-point function which is quadratic in the four-point coupling λ. In order to absorb this into the bare coupling, Z_λ must also contain a term quadratic in λ. Similarly, the diagram



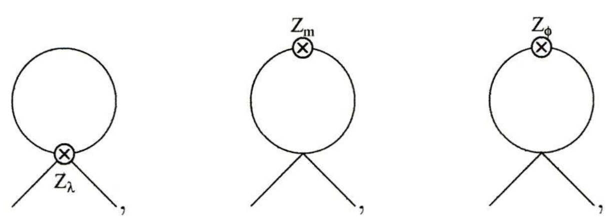
(4.6)

in principle gives contributions to Z_m and Z_ϕ which are linear in λ (although the contribution to Z_ϕ is in fact zero). If we wish to calculate the quadratic term of Z_m, then, there are two types of contribution we must consider. Firstly, there are the two-loop diagrams



(4.7)

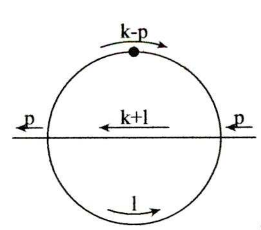
and, secondly, there are one-loop diagrams with factors of the Z's inserted as lower-order corrections to the fields, masses and couplings: [1]



(4.8)

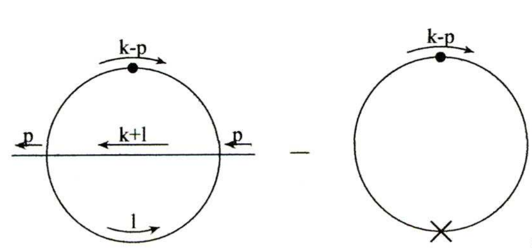
where the $\otimes$'s denote the insertion of a lower-order Z.

So, in the calculation of an amplitude to a given order, we must include contributions from lower orders of the Z's. Whilst the above method demonstrates clearly the ideas involved, an alternative method is employed for ease of computation. These effects of lower-order divergences may be included in an individual multi-loop diagram as counterterms arising from sub-divergences of that diagram. This is equivalent to including the Z's in the manner described above. As an example, in the diagram



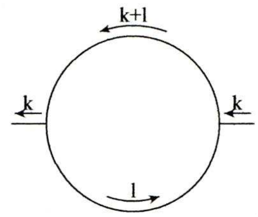
(4.9)

the loop consisting of the l and $k + l$ propagators is independently divergent and its pole term must be subtracted off as



(4.10)

where the $\times$ denotes the pole term of



$$. \tag{4.11}$$

This diagram-by-diagram method of counterterm subtraction is useful as it allows for checking of cancellation of unwanted terms from expansions. This acts as a consistency check during calculation.

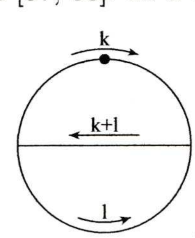
The parameter μ plays the role of renormalisation scale, the scale at which we define our effective action. As the observable parameters of the effective action are obtained from the bare parameters by removal of divergent terms, these parameters are dependent on the renormalisation scale. The β-function, defined by

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}, \tag{4.12}$$

describes the behaviour of the coupling constant g with respect to energy scale. The β-function receives, in principle, simple pole ($\frac{1}{\epsilon}$) contributions from all loop orders. Finiteness of the β-function and μ-independence of the renormalisation constants, however, imply that double pole ($\frac{1}{\epsilon^2}$) and higher terms must cancel from the final expression, leading to consistency conditions for higher-order poles [56].

4.1.2 Infrared Divergences

Since the emission of a particle with infinitesimal momentum is undetectable, the calculation of physical processes should take into account all indistinguishable states, including an infinite series of soft emissions. These contributions are found to cancel with the infrared divergences arising in loop integrals. For this reason, infrared divergences are rather less problematic than ultraviolet when investigating renormalisation but we must still be sure to remove them from our calculations should they arise. As mentioned above, the appearance of an infrared divergence is due to the vanishing of the integrand's denominator, with insufficient powers of momentum in the numerator to render the integral finite. In dimensional regularisation, there is a simple method for the removal of such divergences [57, 58]. In a diagram of the form



$$, \tag{4.13}$$

the k propagator is infrared divergent, having four powers of k in the denominator and only the $\mathrm{d}^d k$ in the numerator to cancel them. In this case, an external momentum p must be fed through this propagator and a correction term added to the corresponding propagator factor in the integral. Thus the diagram in (4.9) would be given by the integral

$$\int \mathrm{d}^d k \left(\frac{1}{(k-p)^4} + \frac{2}{\epsilon}\delta^4(k-p)\right) \int \frac{\mathrm{d}^d l}{l^2(k+l)^2}. \tag{4.14}$$

The additional $\delta^4(k-p)$ term will cancel off the effect of the infrared divergence.

4.1.3 Renormalisation Procedure

In practice, diagrams that can be constructed from the interaction terms of the classical (bare) action may give rise to interaction terms in the renormalised action that were not present in the bare case. Thus, when the classical action undergoes renormalisation, its form is altered. In order to construct an action that is form-invariant under renormalisation, we must include any such additional terms in the full action from the outset. The couplings of these terms may then undergo renormalisation and scaling in the usual way. The couplings assigned to each term must take account of the symmetries of the action. As we wish the symmetries of the classical action to be present after renormalisation but wish to impose no other constraints, terms that transform as a set under any symmetry must have related couplings. Otherwise, we must give independent couplings to independent terms. These rules will allow us to construct general renormalisable actions in the following chapters.

4.2 Renormalisation of Supersymmetric Theories

4.2.1 The Non-renormalisation Theorem

An important result in supersymmetry is the non-renormalisation theorem which states that the holomorphic and antiholomorphic terms of the classical superpotential do not alter under renormalisation [59]. This is due to the fact that these terms are accompanied by only a $\mathrm{d}^2\theta$ or $\mathrm{d}^2\overline{\theta}$ in the action, whilst superfield calculations show that contributions to the effective action must take the form of an integral over $\mathrm{d}^4\theta$ (as appears in the kinetic terms). This is not to say that the masses and couplings themselves are not renormalised, however; merely that the terms in the action suffer no *overall* corrections. That is, we have

$$\begin{aligned} Z_m &= Z_\Phi^{-1}, \\ Z_y &= Z_\Phi^{-\frac{3}{2}}, \end{aligned} \tag{4.15}$$

giving

$$\begin{aligned} \mathcal{L}_{\rm pot} &= \int \mathrm{d}^2\theta \, \left(\frac{1}{2} m_{ijB}\Phi_{iB}\Phi_{jB} + \frac{1}{3} y_{ijkB}\Phi_{iB}\Phi_{jB}\Phi_{kB}\right) \\ &= \int \mathrm{d}^2\theta \, \left(\frac{1}{2} Z_m Z_\Phi m_{ij}\Phi_i\Phi_j + \frac{1}{3} Z_y Z_\Phi^{\frac{3}{2}} y_{ijk}\Phi_i\Phi_j\Phi_k\right) \\ &= \int \mathrm{d}^2\theta \, \left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} y_{ijk}\Phi_i\Phi_j\Phi_k\right). \end{aligned} \tag{4.16}$$

Clearly, this result simplifies loop calculations as, once we have obtained a result for Z_Φ, we immediately have results for Z_m, Z_y also.

4.2.2 Renormalisation in the Uneliminated Formalism

Loop corrections in supersymmetric theories may be calculated in one of three different formalisms. Firstly, there is the superfield formalism in which the superfields themselves are arranged into Feynman graphs, with corresponding superspace Feynman rules. Secondly, there is the eliminated component formalism, in which the auxiliary fields are integrated out of the component action by means of their equations of motion and calculations are performed in the usual manner. It should be noted that the renormalisation constants for the superfield and component formalisms, though related, are not equal. Finally, there is the uneliminated component formalism in which we mainly work. This is almost identical to the eliminated formalism with the addition of auxiliary fields whose propagators are trivial; that is, each auxiliary propagator appearing in a Feynman diagram simply gives a factor of 1 in the corresponding integral. A surprising result in the gauged case, however, is that, in order for the eliminated and uneliminated formalisms to agree, we must introduce non-linear renormalisation of the auxiliary fields, as demonstrated in Ref. [60]. The relation between the bare and renormalised auxiliary fields, analogous to Eq. (4.4), is given by

$$F_{iB} = Z_F^{\frac{1}{2}} F_i + \frac{1}{32\pi^2\epsilon} (\alpha + 3)\, g^2 \left(R^a_{ij} R^a_{jk}\right) \overline{y}_{klm} \overline{\phi}_l \overline{\phi}_m, \tag{4.17}$$

with a similar expression for $\overline{F}_{iB}$.

4.2.3 Dimensional Reduction vs. Dimensional Regularisation

Whilst dimensional regularisation (DREG) has the advantage of being manifestly gauge invariant, it unfortunately has the disadvantage that it explicitly breaks supersymmetry. This is due to the fact that varying the number of spacetime dimensions also varies the number of vector field components, yielding a different number of bosonic and fermionic degrees of freedom. An alternative regularisation method, which is often employed in supersymmetric calculations to avoid this problem, is dimensional reduction (DRED) [61]. In this scheme, the number of dimensions is reduced by compactification, leaving the number of vector field components unaltered. In its original formulation, DRED was found to suffer from inconsistencies regarding relations between ϵ tensors and the metric [62]. These inconsistencies stem from the fact that the non-integral values of d appearing in DREG are in fact properly defined in an infinite-dimensional vector space [63]. In this space, d-dimensional integral calulations are well-defined and obey the appropriate scaling law for a space of d dimensions, including integral values of d. The original form of DRED, on the other hand, was defined in 4 dimensions and was unable to account for non-integral values of d. A consistent form of DRED exists, however, in which the "4-dimensional" space is also realised as an infinite-dimensional space with a scaling law appropriate for four dimensions [64–66], and the equivalence of this scheme to DREG has been demonstrated [67]; results may be converted from one scheme to the other by means of a redefinition of couplings.

Since the differences between DREG and DRED at one loop appear as finite contributions [46] and we are concerned only with the infinite contributions of loop diagrams, our decision to use DREG will not affect the result of the following one-loop calculations. The only calculation in this work which was performed at more than one loop is the two-loop calculation of chapter 6; however, as the subject of this calculation is the deformed Wess-Zumino model, there are no gauge multiplets in which the difference between DRED and DREG may arise. Therefore, no difficulties with our non-supersymmetric choice of regularisation scheme will be encountered in the following chapters, although we will return to this matter in chapter 9 when we discuss higher-order calculations.

4.3 Renormalisation of $\mathcal{N} = \frac{1}{2}$ Theories

Due to the lack of Hermiticity in non-anticommutative theories, we need not renormalise chiral and antichiral fields in the same manner. In fact, we find that the F fields renormalise as in the undeformed case whilst the bare $\overline{F}$ fields gain additional C-dependent divergent terms [51]:

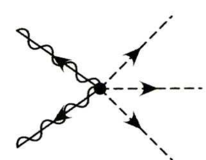
$$\begin{aligned} F_{iB} &= Z_F^{\frac{1}{2}} F_i + \frac{1}{32\pi^2\epsilon} (\alpha + 3)\, g^2 \left(R^a_{ij} R^a_{jk}\right) \overline{y}_{klm} \overline{\phi}_l \overline{\phi}_m, \\ \overline{F}_{iB} &= Z_F^{\frac{1}{2}} \overline{F}_i + \frac{1}{32\pi^2\epsilon} (\alpha + 3)\, g^2 \left(R^a_{ij} R^a_{jk}\right) y_{klm} \phi_l \phi_m + C\text{–dependent terms.} \end{aligned} \tag{4.18}$$

In addition to this, it has been found that the gaugino must also undergo a non-linear renormalisation [50] including C-dependent divergent terms.

The lack of hermiticity is also crucial for renormalisability in $\mathcal{N} = \frac{1}{2}$ theories. The action of a non-anticommutative theory generally contains interaction terms with large numbers of legs which would appear to lead to a non-renormalisable theory; for instance, as we shall see in chapter 7, the $U(1)$ model contains the term

$$\mathcal{L}_{\mathrm{U}(1)} \supset |C|^2 \overline{\phi}^a \overline{\phi}^b \overline{\phi}^c \overline{\lambda}\overline{\lambda}, \tag{4.19}$$

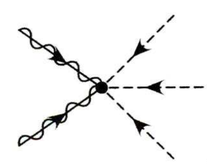
corresponding to the vertex



(4.20)

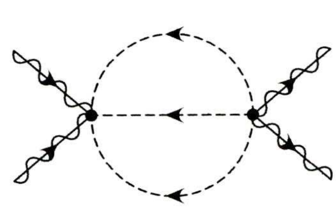
which, having a coupling of negative mass dimension, would normally suggest non-renormalisability. However, as a propagator connects barred to unbarred fields, in order to construct the divergent graphs that would lead to non-renormalisability, we would require the conjugate term

$$|C|^2 \phi^a \phi^b \phi^c \lambda\lambda \quad \Longrightarrow$$



, (4.21)

which is missing due to non-Hermiticity [68]. That is: we cannot construct, for example, the graph



. (4.22)

Under non-anticommutative deformation of the action, the form of the holomorphic part of the superpotential is altered radically enough to destroy the non-renormalisation property of §4.2.1. However, the anti-holomorphic part is sufficiently similar to the non-deformed case that these terms still suffer no renormalisation [69], a result that will be of use in later chapters.

Chapter 5

The Non-anticommutative Non-linear σ-model

In this chapter, we discuss the deformation of the two-dimensional $\mathcal{N} = 2$ supersymmetric non-linear σ-model [70]. We begin with a brief discussion of non-linear σ-models in general before progressing, in §5.2, to a more detailed treatment of the (undeformed) $\mathcal{N} = 2$ supersymmetric case. We introduce differential operators which are able to implement the supersymmetry transformations in terms of the component fields, and discuss a concise method for expressing the one-loop divergences of the theory in terms of these operators. In §5.3, we show how these differential operators may be extended to the deformed case and finally, in §5.4, we investigate the effects of the deformation on the renormalisation of the model.

5.1 The Non-Linear σ-model

In the non-supersymmetric case, a non-linear σ-model consists of a set of scalar fields ϕ^i mapped from a base space to a target manifold by a Lagrangian of the form

$$\mathcal{L} = g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j. \tag{5.1}$$

Clearly, the coefficients $g_{ij}(\phi)$ must form the components of a symmetric tensor and, in fact, they act as a Riemannian metric on the target manifold [71]. It is also possible, therefore, to calculate the Riemann and Ricci tensors, R^i_{jkl}, R_{ij}, for the target space. The non-linear σ-model is non-renormalisable in any dimension higher than two, whilst in the two-dimensional case, the dimensionlessness of the fields allows interaction terms of arbitrarily high order to be included in the form of the ϕ-dependence of the metric. However, though the two-dimensional model is power-counting renormalisable, there is an additional subtlety: as $g_{ij}(\phi)$ can take any form in this case, there are infinitely many terms that may be added. For renormalisability, these possible terms must be constrained by imposing, for instance, an $O(N)$ symmetry on the target space (where N is the dimension of the target space or, equivalently, the number of fields in the model) [1, 72]. The two-dimensional model has been found to be of importance in String Theory, as its domain is then the two-dimensional worldsheet. In the supersymmetric case, the fields are replaced with chiral superfields. As spinors in a four-dimensional space have two components whilst those in two-dimensional space have only one, the reduction of a four-dimensional theory to two dimensions will double the number of supercharges. In order to coincide with σ-models derived via dimensional reduction, we take $\mathcal{N} = 2$ for our undeformed model. With $\mathcal{N} = 2$ supersymmetry, the chiral superfields, and hence the target manifold, are complex; in fact, the geometry of the target space is found to be a Kähler geometry [73, 74]. That is, the target space has the structure of both a complex Riemannian and a symplectic manifold, with compatible symmetric and antisymmetric bilinear forms. In this case, the metric is expressible in terms of a potential function, K, known as the Kähler potential [75]:

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial z^{\bar{j}}}, \tag{5.2}$$

where z^i are the complex coordinates of the manifold.

The undeformed $\mathcal{N} = 2$ Kähler σ-model and its renormalisation have been studied extensively in the context of String Theory. It was thought for a while that its only divergences were at the one-loop level, where they can be interpreted as a correction to the Kähler metric in the form of the Ricci tensor, until explicit calculations [76, 77] revealed a divergence at the four-loop level. The undeformed classical component action may be expressed in the form of all four supercharges acting on the Kähler potential, and the one-loop corrections may be written in an equally concise form as will be discussed in the following section. If one writes down differential operators representing the supercharges, but which act directly on the component fields, the action may also be expressed in terms of these.

Non-anticommutative versions of particular non-linear σ-models, namely those with a CP^N target space, have been constructed by dimensional reduction from four dimensions [78], and the one-loop corrections computed [79]. The CP^N models are of particular interest as the two-dimensional case shares a number of poperties with four-dimensional supersymmetric gauge theory, such as asymptotic freedom and instanton effects [80]. A non-anticommutative version of the general $\mathcal{N} = 2$ Kähler σ-model has also been constructed directly in two dimensions, initially in Refs. [81, 82] but then given an elegant reformulation in Refs. [83, 84]. We predominantly follow the notation of Ref. [83], where the deformation was interpreted as a "smearing" of the Kähler geometry; that is, correction terms to the Kähler potential appear as a power series in the non-anticommutativity parameter.

The motivation for the work presented in this chapter was to investigate whether the one-loop corrections in the deformed theory would exhibit a similar "smearing" as in the classical theory as presented in [83]. It turns out that the number of one-loop diagrams in the deformed theory is rather large, at least in the component formulation in which we work. However, as in the undeformed case, they can be expressed in terms of differential operators implementing the undeformed supersymmetry generators $Q_\pm$ (using light-cone coordinates in two dimensions), acting on a simpler "kernel", $\mathcal{K}$. In an attempt to write this kernel in an even more compact form in terms of the remaining supercharges, we find expressions for differential operators implementing the deformed supersymmetry generators $\overline{Q}_\pm$. The construction of these operators is given in §5.3. However, having given an expression for the kernel in §5.4, we then demonstrate that it is unfortunately impossible to write $\mathcal{K}$ in a shorter form using the operators representing $\overline{Q}_\pm$.

5.2 The $\mathcal{N} = 2$ Non-linear σ-Model

In this section we set the scene for the analysis by describing in detail the case of undeformed supersymmetry in two dimensions, focusing on the use of differential operators to implement the supersymmetry and simplify the description as our goal is to introduce similar operators in the deformed case. As the supercharges in two dimensions act independently on left- and right-moving components, it is convenient in this case to use "lightcone" coordinates $x^\pm$, $\theta^\pm$, $\overline{\theta}^\pm$ and so the identities of the previous chapters will be rewritten in this formalism. The expression "lightcone coordinates" strictly applies only to Minkowski spacetime and in the non-anticommutative case we are obliged to work in a spacetime of Euclidean signature. However, following the conventions of [83], our coordinates are constructed in a manner analogous to that in the Minkowski case, as $x_\pm \equiv \frac{1}{2}(x^0 \pm x^1)$.

We now consider a theory with a multiplet of chiral superfields $\Phi^i\left(x^\pm, \theta^\pm, \overline{\theta}^\pm\right)$ (with components ϕ^i, ψ^i, F^i) and their conjugate fields $\overline{\Phi}^{\overline{i}}$, $\overline{\phi}^{\overline{i}}$, $\overline{\psi}^{\overline{i}}$, $\overline{F}^{\overline{i}}$. The $\mathcal{N}=2$ non-linear σ-model action is then given by

$$S_0 = \int \mathrm{d}^2x \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} K\left(\Phi, \overline{\Phi}\right). \tag{5.3}$$

In the two-dimensional case, the supercharges are given by

$$\begin{aligned} Q_\pm &= \frac{\partial}{\partial\theta^\pm}, \\ \overline{Q}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\frac{\partial}{\partial y^\pm}, \end{aligned} \tag{5.4}$$

where

$$y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm \tag{5.5}$$

are the chiral coordinates in two dimensions.

These charges now satisfy the algebra

$$\begin{aligned} Q_\pm^2 &= \overline{Q}_\pm^2 = 0, \\ \{Q_+, Q_-\} &= \{\overline{Q}_+, \overline{Q}_-\} = 0, \\ \{\overline{Q}_\pm, Q_\pm\} &= -i\partial_\pm, \end{aligned} \tag{5.6}$$

and the superfields have the component field expansions

$$\begin{aligned} \Phi &= \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F, \\ \overline{\Phi} &= \overline{\phi} + \overline{\theta}^+\left[\overline{\psi}_+ - i\theta^+\partial_+\overline{\phi}\right] + \overline{\theta}^-\left[\overline{\psi}_- - i\theta^-\partial_-\overline{\phi}\right] \\ &\quad + \overline{\theta}^+\overline{\theta}^-\left[\overline{F} + i\theta^+\partial_+\overline{\psi}_- - i\theta^-\partial_-\overline{\psi}_+ + \theta^+\theta^-\partial_+\partial_-\overline{\phi}\right], \end{aligned} \tag{5.7}$$

where the component fields are functions of $y^\pm$, as defined in Eq. (5.5). As the supercharges operate on the superspace coordinates, their effect on an individual component field is not immediately obvious without reference to the superfield. To account for this, it is useful to represent the effect of the charges $Q_\pm, \overline{Q}_\pm$ by differential operators $q_\pm, \overline{q}_\pm^0$ acting directly on the fields, according to

$$\begin{aligned} \left[Q_\pm, \Phi\right] &= q_\pm\Phi, \\ \left[\overline{Q}_\pm, \Phi\right] &= \overline{q}_\pm^0\Phi, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} q_\pm &= \psi_\pm\frac{\partial}{\partial\phi} \mp F\frac{\partial}{\partial\psi_\mp} - i\partial_\pm\overline{\phi}\frac{\partial}{\partial\overline{\psi}_\pm} \pm i\partial_\pm\overline{\psi}_\mp\frac{\partial}{\partial\overline{F}}, \\ \overline{q}_\pm^0 &= -\overline{\psi}_\pm\frac{\partial}{\partial\overline{\phi}} \pm \overline{F}\frac{\partial}{\partial\overline{\psi}_\mp} + i\partial_\pm\phi\frac{\partial}{\partial\psi_\pm} \mp i\partial_\pm\psi_\mp\frac{\partial}{\partial F}. \end{aligned} \tag{5.9}$$

We use the superscript "0" to denote the undeformed case. Since $q_\pm$ will be unchanged in the deformed case, no superscript is needed for the unbarred operators. These operators have anticommutation properties analogous to Eq. (5.6), except that

$$\begin{aligned} \{\overline{q}^0_+, q_+\} &= i\partial_+, \\ \{\overline{q}^0_-, q_-\} &= i\partial_-, \end{aligned} \tag{5.10}$$

where the change in sign is due to the definition of the differential operators in terms of commutators of the supercharges as can be seen by commuting Eq. (5.8) with $\overline{Q}_\pm, Q_\pm$ respectively and using

$$\left[q_\pm, \overline{Q}_\pm\right] = \left[\overline{q}^0_\pm, Q_\pm\right] = 0, \tag{5.11}$$

which follows from

$$\left[q_\pm, \partial_\pm\right] = \left[\overline{q}^0_\pm, \partial_\pm\right] = 0 \tag{5.12}$$

in conjunction with Eq. (5.6) and the generalised Jacobi identity:

$$[A, [B, C]] + [B, [A, C]] = [\{A, B\}, C]. \tag{5.13}$$

The transformations of $\Phi, \overline{\Phi}$ induced by $\delta_\epsilon = \epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-$ are given by

$$\begin{aligned} \delta_\epsilon \Phi &= \left[\epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-, \Phi\right], \\ \delta_\epsilon \overline{\Phi} &= \left[\epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-, \overline{\Phi}\right] \end{aligned} \tag{5.14}$$

so, combining Eq. (5.4) with Eq. (5.7), we find the following transformations of the component fields:

$$\begin{aligned} \delta_\epsilon \phi &= \epsilon^+ \psi_+ + \epsilon^- \psi_-, \\ \delta_\epsilon \psi_+ &= \epsilon^- F + i\overline{\epsilon}^+ \partial_+ \phi, \\ \delta_\epsilon \psi_- &= -\epsilon^+ F + i\overline{\epsilon}^- \partial_- \phi, \\ \delta_\epsilon F &= -i\overline{\epsilon}^+ \partial_+ \psi_- + i\overline{\epsilon}^- \partial_- \psi_+, \\ \delta_\epsilon \overline{\phi} &= -\overline{\epsilon}^+ \overline{\psi}_+ - \overline{\epsilon}^- \overline{\psi}_-, \\ \delta_\epsilon \overline{\psi}_+ &= -i\epsilon^+ \partial_+ \overline{\phi} - \overline{\epsilon}^- \overline{F}, \\ \delta_\epsilon \overline{\psi}_- &= -i\epsilon^- \partial_- \overline{\phi} + \overline{\epsilon}^+ \overline{F}, \\ \delta_\epsilon \overline{F} &= i\epsilon^+ \partial_+ \overline{\psi}_- - i\epsilon^- \partial_- \overline{\psi}_+. \end{aligned} \tag{5.15}$$

Using Eqs. (5.7), (5.8), we can also write the transformations in terms of the differential operators:

$$\delta_\epsilon \phi = \left(\epsilon^+ q_+ + \epsilon^- q_- + \overline{\epsilon}^+ \overline{q}_+ + \overline{\epsilon}^- \overline{q}_-\right) \phi, \tag{5.16}$$

with similar expressions for the other component fields.

The effect of the $\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}$ in Eq. (5.3) is to extract the $\theta^2\overline{\theta}^2$ term from the expansion of $K\left(\Phi, \overline{\Phi}\right)$ giving the action in terms of components as

$$\begin{aligned} S_0 = \int \mathrm{d}^2 x \Big[& K_{\overline{j}} \partial_+ \partial_- \overline{\phi}^{\overline{j}} + K_{\overline{j}\overline{k}} \partial_+ \overline{\phi}^{\overline{j}} \partial_- \overline{\phi}^{\overline{k}} \\ & + K_{i\overline{j}} \left(i\psi^i_+ \partial_- \overline{\psi}^{\overline{j}}_+ + i\psi^i_- \partial_+ \overline{\psi}^{\overline{j}}_- + F^i \overline{F}^{\overline{j}} \right) \\ & - K_{ik\overline{j}} \psi^i_+ \psi^k_- \overline{F}^{\overline{j}} - K_{\overline{i}\overline{k}j} \overline{\psi}^{\overline{i}}_+ \overline{\psi}^{\overline{k}}_- F^j \\ & + iK_{i\overline{j}\overline{k}} \left(\psi^i_+ \overline{\psi}^{\overline{j}}_+ \partial_- \overline{\phi}^{\overline{k}} + \psi^i \overline{\psi}^{\overline{j}}_- \partial_+ \overline{\phi}^{\overline{k}} \right) \\ & + K_{ij\overline{i}\overline{j}} \psi^i_+ \psi^j_- \overline{\psi}^{\overline{i}}_+ \overline{\psi}^{\overline{j}}_- \Big], \end{aligned} \tag{5.17}$$

where

$$K_i \equiv \frac{\partial K}{\partial \phi^i}. \tag{5.18}$$

Applying the differential operators, Eq. (5.9), to the action, Eq. (5.17), we may verify that

$$q_\pm S_0 = \overline{q}^0_\pm S_0 = 0, \tag{5.19}$$

demonstrating the invariance of the action under supersymmetry transformations, according to Eq. (5.16).

The action in Eq. (5.17) can also be written using the differential operators $q_\pm, \overline{q}^0_\pm$ as

$$S_0 = \int \mathrm{d}^2 x q_- q_+ \overline{q}^0_- \overline{q}^0_+ K, \tag{5.20}$$

which guarantees Eq. (5.19) due to the nilpotency of $q_\pm, \overline{q}^0_\pm$, which in turn follows from that of $Q_\pm, \overline{Q}_\pm$ in Eq. (5.6). We may also eliminate the auxiliary fields $F, \overline{F}$ using their equations of motion to rewrite the action in the form

$$\begin{aligned} S_0 = \int \mathrm{d}^2 x \Big[& g_{i\bar{j}} \left(\partial_+ \phi^i \partial_- \overline{\phi}^{\bar{j}} + i\psi^i_+ \partial_- \overline{\psi}^{\bar{j}}_+ + i\psi^i_- \partial_+ \overline{\psi}^{\bar{j}} \right) \\ & + iR_{i\bar{i}j\bar{j}} \psi^i_+ \psi^j_- \overline{\psi}^{\bar{i}}_+ \overline{\psi}^{\bar{j}}_- \Big], \end{aligned} \tag{5.21}$$

where $R_{i\bar{i}j\bar{j}}$ is the Riemann curvature tensor constructed from the Kähler metric $g_{i\bar{j}} \equiv K_{i\bar{j}}$. By writing the action in this form we see that it is expressible in terms of the metric.

At the quantum level the renormalisation of the model may be achieved by replacing the classical Kähler potential by a bare version, K_B, chosen so as to cancel the ultraviolet divergences order by order. Using dimensional regularisation with the spacetime dimension continued to $d = 2 - \epsilon$, at one loop we have

$$K_B = K + \frac{1}{2\pi\epsilon} \mathrm{tr} \left(\ln K_{i\bar{j}} \right), \tag{5.22}$$

which corresponds to replacing the Kähler metric by

$$g_{Bi\bar{j}} = g_{i\bar{j}} + \frac{1}{2\pi\epsilon} R_{i\bar{j}}, \tag{5.23}$$

where $R_{i\bar{j}}$ is the Ricci tensor [85]. No higher-order divergences appear until the four-loop level.

Just as the classical action may be obtained by the operators $q_\pm, \overline{q}^0_\pm$ acting on K as in Eq. (5.20), we may write the bare action as

$$S_{0B} = \int \mathrm{d}^2 x q_- q_+ \overline{q}^0_- \overline{q}^0_+ K_B. \tag{5.24}$$

By comparing this with Eq. (5.22) we see that the expression

$$q_- q_+ \overline{q}^0_- \overline{q}^0_+ \mathrm{tr} \ln K_{i\bar{j}} \tag{5.25}$$

has the effect of reproducing the one-loop divergences in a compact form.

5.3 Non-anticommutative supersymmetry in two dimensions

In this section we repeat the analysis of the previous section for the case of deformed two-dimensional supersymmetry. We take the deformation to be of the form

$$\begin{aligned} \left(\theta^{\pm}\right)^2 = \left(\overline{\theta}^{\pm}\right)^2 &= 0, \\ \{\overline{\theta}^{+}, \overline{\theta}^{-}\} &= 0, \\ \{\theta^{+}, \theta^{-}\} &= \frac{1}{M}. \end{aligned} \tag{5.26}$$

This form for the non-anticommutativity parameter allows us to interpret M as the energy-scale of the deformation.

The supercharges then satisfy the algebra

$$\begin{aligned} Q_{\pm}^2 = \overline{Q}_{\pm}^2 &= 0, \\ \{Q_{+}, Q_{-}\} &= 0, \\ \{\overline{Q}_{+}, \overline{Q}_{-}\} &= -\frac{4}{M}\frac{\partial^2}{\partial y^{+}\partial y^{-}}, \\ \{\overline{Q}_{\pm}, Q_{\pm}\} &= -i\partial_{\pm}, \end{aligned} \tag{5.27}$$

and the star-product in this case may be expressed as

$$\begin{aligned} \theta^{+} \star \theta^{-} &= \theta^{+}\theta^{-} + \frac{1}{2M}, \\ \theta^{-} \star \theta^{+} &= -\theta^{+}\theta^{-} + \frac{1}{2M}, \\ \theta^{+} \star \theta^{+}\theta^{-} &= -\frac{1}{2M}\theta^{+}, \\ \theta^{-} \star \theta^{+}\theta^{-} &= \frac{1}{2M}\theta^{-}, \\ \theta^{+}\theta^{-} \star \theta^{+}\theta^{-} &= \frac{1}{4M^2}. \end{aligned} \tag{5.28}$$

We now wish to construct differential operators $\overline{q}_{\pm}$ representing the effects of $\overline{Q}_{\pm}$ in the deformed case in a similar manner to Eq. (5.8), extending the $\overline{q}_{\pm}^{0}$ given in Eq. (5.9) for the undeformed case. It should be noted that the differential operators $q_{\pm}$ are not affected by the deformation. We begin by examining the effects of $\overline{Q}_{\pm}$ on powers of Φ alone, since mixed products of Φ and $\overline{\Phi}$ present additional complications.

Defining

$$I_r^{(n)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\xi \left(\frac{\xi}{M}\right)^r \left(\phi + \frac{\xi}{M}F\right)^n, \tag{5.29}$$

we may show, using the methods of [83], that

$$\Phi_{\star}^{n} = \left(1 + \theta^{+}q_{+}\right)\left(1 + \theta^{-}q_{-}\right)\left(I_0^{(n)} - q_{+}q_{-}I_1^{(n)}\right), \tag{5.30}$$

where $\Phi_{\star}^{n}$ denotes the star-product of n copies of Φ. When acting on $\Phi_{\star}^{n}$, the $\overline{Q}_{\pm}$ are represented by the differential operators

$$\begin{aligned} \overline{q}_{+}^{\Phi} &= \overline{q}_{+}^{0} - \frac{i}{2M}\partial_{+}q_{-} + i\left(-q'_{+}q'_{-}\left[\partial'_{+}q'_{-}\right]\widetilde{\mathcal{O}} + \partial'_{+}q'_{-}\mathcal{O} + \left[\partial'_{+}q'_{-}\right]\mathcal{O}\right), \\ \overline{q}_{-}^{\Phi} &= \overline{q}_{-}^{0} - \frac{i}{2M}\partial_{-}q_{+} - i\left(-q'_{+}q'_{-}\left[\partial'_{-}q'_{+}\right]\widetilde{\mathcal{O}} + \partial'_{-}q'_{+}\mathcal{O} + \left[\partial'_{-}q'_{+}\right]\mathcal{O}\right), \end{aligned} \tag{5.31}$$

where a prime denotes the part of the operator containing only derivatives with respect to the chiral (but not the anti-chiral) fields such that, in particular

$$\partial'_{\pm} = \partial_{\pm}\phi\frac{\partial}{\partial\phi} + \partial_{\pm}\psi_{+}\frac{\partial}{\partial\psi_{+}} + \partial_{\pm}\psi_{-}\frac{\partial}{\partial\psi_{-}} + \partial_{\pm}F\frac{\partial}{\partial F}. \tag{5.32}$$

Furthermore, $\left[\partial'_{+}q'_{-}\right]$ above is defined by

$$\left[\partial'_{+}q'_{-}\right] = \partial_{+}\psi_{-}\frac{\partial}{\partial\phi} + \partial_{+}F\frac{\partial}{\partial F} \tag{5.33}$$

and $\mathcal{O}, \widetilde{\mathcal{O}}$ are required to obey

$$\begin{aligned} \mathcal{O}I_0^{(n)} &= I_1^{(n)}, \\ \mathcal{O}I_1^{(n)} &= I_2^{(n)} - \widetilde{\mathcal{O}}I_0^{(n)}. \end{aligned} \tag{5.34}$$

These properties are guaranteed by the following definitions:

$$\begin{aligned} \mathcal{O} &= \sum_{r=1}^{\infty} a_r \left(\frac{1}{M^2}\right)^r \left(F\frac{\partial}{\partial\phi}\right)^{2r-1}, \\ \widetilde{\mathcal{O}} &= \sum_{r=1}^{\infty} (2r-1)\, a_r \left(\frac{1}{M^2}\right)^r \left(F\frac{\partial}{\partial\phi}\right)^{2r-2}, \end{aligned} \tag{5.35}$$

where the a_r must satisfy, for each $n \geq 1$,

$$\sum_{r=0}^{n-1} \frac{a_{n-r}}{2^{2r}\,(2r+1)\,(2r)!} = \frac{1}{2^{2n}\,(2n+1)\,(2n-1)}. \tag{5.36}$$

Whilst there appears to be no closed expression for the a_r satisfying this property, they may be calculated iteratively and the first terms in the sequence are

$$a_1 = \frac{1}{12}, \qquad a_2 = -\frac{1}{720}, \qquad a_3 = \frac{1}{2^5\cdot 3^3\cdot 5\cdot 7}. \tag{5.37}$$

The following relations follow directly from the undeformed differential operators, Eq. (5.9), and the definition of I, Eq. (5.29):

$$\begin{aligned} \overline{q}^{0\prime}_{+}I_r^{(n)} &= -i\left[\partial'_{+}q'_{-}\right]I_{r+1}^{(n)}, \\ \overline{q}^{0\prime}_{-}I_r^{(n)} &= i\left[\partial'_{-}q'_{+}\right]I_{r+1}^{(n)}, \\ q''_{+}I_0^{(n)} &= q''_{-}I_0^{(n)} = 0, \\ q''_{+}I_1^{(n)} &= q''_{-}I_1^{(n)} = 0, \end{aligned} \tag{5.38}$$

where a double prime denotes the part of the operator containing derivatives with respect to the anti-chiral (but not the chiral fields).

We may use these relations to verify that the operators in Eq. (5.31) do indeed represent the operators $\overline{Q}_{\pm}$ according to

$$\left[\overline{Q}_{\pm}, \Phi^n_{\star}\right] = \overline{q}^{\Phi}_{\pm}\Phi^n_{\star}. \tag{5.39}$$

It is also straightforward to check that the operators in Eq. (5.31) satisfy the anti-commutation relations of Eq. (5.27) using

$$\left[\overline{q}^0_\pm, F\frac{\partial}{\partial\phi}\right] = \mp i\left[\partial'_\pm q'_\mp\right], \tag{5.40}$$

which follows by direct computation and which implies

$$\left[\overline{q}^0_\pm, \mathcal{O}\right] = \mp i\left[\partial'_\pm q'_\mp\right]\widetilde{\mathcal{O}}. \tag{5.41}$$

When acting on products of both Φ and $\overline{\Phi}$, the situation is more complicated and the operators representing $\overline{Q}_\pm$ will require further modification. We have $\overline{\Phi}^n_\star = \overline{\Phi}^n$ and we find

$$\begin{aligned}\Phi^n_\star \star \overline{\Phi}^m = &\left(1+\theta^+ q_+\right)\left(1+\theta^- q_-\right)\\ &\times\left[1-\overline{\theta}^+\left(\overline{q}^{0\prime\prime}_+ - \frac{i}{2M}\partial''_+ q'_-\right)\right]\left[1-\overline{\theta}^-\left(\overline{q}^{0\prime\prime}_- - \frac{i}{2M}\partial''_- q'_+\right)\right]\\ &\times\left(I^{(n)}_0 - q_+q_- I^{(n)}_1\right)\overline{\phi}^m,\end{aligned} \tag{5.42}$$

in which case we have

$$\begin{aligned}\left[\overline{Q}_+, \Phi^n_\star \star \overline{\Phi}^m\right]_\star &= \left\{\overline{q}^\Phi_+ - \frac{i}{2M}\left(\partial''_+ q'_- - \partial'_+ q''_-\right)\right\}\Phi^n_\star \star \overline{\Phi}^m,\\ \left[\overline{Q}_-, \Phi^n_\star \star \overline{\Phi}^m\right]_\star &= \left\{\overline{q}^\Phi_- - \frac{i}{2M}\left(\partial''_- q'_+ - \partial'_- q''_+\right)\right\}\Phi^n_\star \star \overline{\Phi}^m.\end{aligned} \tag{5.43}$$

However, if we begin with

$$\begin{aligned}\overline{\Phi}^m \star \Phi^n_\star = &\left(1+\theta^+ q_+\right)\left(1+\theta^- q_-\right)\\ &\times\left[1-\overline{\theta}^+\left(\overline{q}^{0\prime\prime}_+ + \frac{i}{2M}\partial''_+ q'_-\right)\right]\left[1-\overline{\theta}^-\left(\overline{q}^{0\prime\prime}_- + \frac{i}{2M}\partial''_- q'_+\right)\right]\\ &\times\left(I^{(n)}_0 - q_+q_- I^{(n)}_1\right)\overline{\phi}^m,\end{aligned} \tag{5.44}$$

then we have

$$\begin{aligned}\left[\overline{Q}_+, \overline{\Phi}^m \star \Phi^n_\star\right]_\star &= \left\{\overline{q}^\Phi_+ + \frac{i}{2M}\left(\partial''_+ q'_- - \partial'_+ q''_-\right)\right\}\overline{\Phi}^m \star \Phi^n_\star,\\ \left[\overline{Q}_-, \overline{\Phi}^m \star \Phi^n_\star\right]_\star &= \left\{\overline{q}^\Phi_- + \frac{i}{2M}\left(\partial''_- q'_+ - \partial'_- q''_+\right)\right\}\overline{\Phi}^m \star \Phi^n_\star.\end{aligned} \tag{5.45}$$

We see from these relations that the operators representing $\overline{Q}_\pm$ are modified in different ways depending on whether they act on $\Phi^n_\star \star \overline{\Phi}^m$ or on $\overline{\Phi}^m \star \Phi^n_\star$. It is unusual to find that the representation of the operator depends on the ordering of the term on which it acts. Fortunately, however, we are only interested in the deformed version of the Kähler potential, in which, for naturalness, each term should be defined as a symmetrised star-product of Φ's and $\overline{\Phi}$'s, and therefore the ordering problem will not arise. For such a symmetrised product, the representations of $\overline{Q}_\pm$ will again be different from those given in Eq. (5.31) and indeed from those given in Eqs. (5.43), (5.45). For an undeformed Kähler potential

$$K\left[\Phi, \overline{\Phi}\right] = \sum_{n,m} K_{n,m}\Phi^n\overline{\Phi}^m, \tag{5.46}$$

the equivalent deformed potential is given by

$$K_\star\left[\Phi,\overline{\Phi}\right]=\sum_{n,m}\left[\Phi^n\overline{\Phi}^m\right]_\star, \tag{5.47}$$

where $\left[\Phi^n\overline{\Phi}^m\right]_\star$ represents the symmetrised star-product of n Φ's and m $\overline{\Phi}$'s. It may be shown that

$$\begin{aligned}K_\star\left[\Phi,\overline{\Phi}\right] = {} & \left(1+\theta^+q_+\right)\left(1+\theta^-q_-\right)\left(1-\overline{\theta}^+\overline{q}_+^{0\prime\prime}\right)\left(1-\overline{\theta}^-\overline{q}_-^{0\prime\prime}\right)\\ & \times\left[K_0\left(\phi,F,\overline{\phi}\right)-q_+q_-K_1\left(\phi,F,\overline{\phi}\right)\right]\\ & -\frac{1}{4M^2}\overline{\theta}^+\overline{\theta}^-q'_+q'_-\partial''_+\partial''_-K_0\left(\phi,F,\overline{\phi}\right),\end{aligned} \tag{5.48}$$

where

$$K_m\left(\phi,F,\overline{\phi}\right)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}\xi\xi^mK\left(\phi+\frac{\xi}{M}F,\overline{\phi}\right). \tag{5.49}$$

The symmetrisation has resulted in the disappearance of most of the terms involving a factor of $\frac{1}{2M}$ in Eqs. (5.42), (5.44). Correspondingly, we no longer need to include the $\frac{1}{2M}$ terms of Eqs. (5.43), (5.45) in our operators. However, the residual $\frac{1}{4M^2}$ term requires a modification of the operators given in Eqs. (5.31), so that

$$\begin{aligned}\overline{q}_+ = {} & \overline{q}_+^0-\frac{i}{2M}\partial_+q_--\frac{i}{4M^2}\left(\partial''_+q'_+q'_-q''_-+\partial'_+q'_-q''_+q''_-\right)\\ & +i\left(-q'_+q'_-\left[\partial'_+q'_-\right]\widetilde{\mathcal{O}}+\partial'_+q'_-\mathcal{O}+\left[\partial'_+q'_-\right]\mathcal{O}\right),\\ \overline{q}_- = {} & \overline{q}_+^0-\frac{i}{2M}\partial_-q_++\frac{i}{4M^2}\left(-\partial''_-q'_+q'_-q''_++\partial'_-q'_+q''_+q''_-\right)\\ & -i\left(-q'_+q'_-\left[\partial'_-q'_+\right]\widetilde{\mathcal{O}}+\partial'_-q'_+\mathcal{O}+\left[\partial'_-q'_+\right]\mathcal{O}\right).\end{aligned} \tag{5.50}$$

We can verify that these operators do indeed implement the operators $\overline{Q}_\pm$ according to

$$\left[\overline{Q}_\pm,K_\star\right]_\star=\overline{q}_\pm K_\star \tag{5.51}$$

by using the analogue of Eq. (5.34) for the Kähler potential,

$$\begin{aligned}\mathcal{O}K_0&=K_1,\\ \mathcal{O}K_1&=K_2-\widetilde{\mathcal{O}}K_0\end{aligned} \tag{5.52}$$

along with the analogue of Eq. (5.38),

$$\begin{aligned}\overline{q}'_+K_r&=-i\left[\partial'_+q'_-\right]K_{r+1},\\ \overline{q}'_-K_r&=i\left[\partial'_-q'_+\right]K_{r+1},\\ q''_+K_0&=q''_-K_0=0,\\ q''_+K_1&=q''_-K_1=0.\end{aligned} \tag{5.53}$$

The action is given by the $\theta^2\overline{\theta}^2$ term and hence, from Eq. (5.48), we have

$$S=\int\mathrm{d}^2xq_-q_+\overline{q}''_-\overline{q}''_+\left(K_0-q_+q_-K_1\right), \tag{5.54}$$

which can be expanded as [81–83]

$$
\begin{aligned}
S = \int \mathrm{d}^2x \Big\{ & \partial_{\bar{j}} K_0 \partial_+ \partial_- \overline{\phi}^{\bar{j}} + \partial_{\bar{j}} \partial_{\bar{k}} K_0 \partial_+ \overline{\phi}^{\bar{j}} \partial_- \overline{\phi}^{\bar{k}} \\
& + \partial_i \partial_{\bar{j}} K_0 \left(i\psi_+^i \partial_- \overline{\psi}_+^{\bar{j}} + i\psi_-^i \partial_+ \overline{\psi}_-^{\bar{j}} + F^i \overline{F}^{\bar{j}} \right) \\
& - \partial_i \partial_k \partial_{\bar{j}} K_0 \psi_+^i \psi_-^k \overline{F}^{\bar{j}} - \partial_{\bar{i}} \partial_{\bar{k}} \partial_j K_0 \overline{\psi}_+^{\bar{i}} \overline{\psi}_-^{\bar{k}} F^j \\
& + i \partial_i \partial_{\bar{j}} \partial_{\bar{k}} K_0 \left(\psi_+^i \overline{\psi}_+^{\bar{j}} \partial_- \overline{\phi}^{\bar{k}} + \psi_-^i \overline{\psi}_-^{\bar{j}} \partial_+ \overline{\phi}^{\bar{k}} \right) \\
& + \partial_i \partial_{\bar{j}} \partial_{\bar{i}} \partial_{\bar{j}} K_0 \psi_+^i \psi_-^j \overline{\psi}_+^{\bar{i}} \partial \psi_-^{\bar{j}} \\
& + \frac{1}{M} \Big(\partial_i \partial_{\bar{j}} K_1 F^i \partial_+ \partial_- \overline{\phi}^{\bar{j}} - \partial_i \partial_k \partial_{\bar{j}} K_1 \psi_+^i \psi_-^k \partial_+ \partial_- \overline{\phi}^{\bar{j}} \\
& + \partial_i \partial_{\bar{j}} \partial_{\bar{k}} K_1 F^i \partial_+ \overline{\phi}^{\bar{j}} \partial_- \overline{\phi}^{\bar{k}} - \partial_i \partial_k \partial_{\bar{j}} \partial_{\bar{k}} K_1 \psi_+^i \psi_-^k \partial_+ \overline{\phi}^{\bar{j}} \partial_- \overline{\phi}^{\bar{k}} \Big) \Big\}.
\end{aligned}
\tag{5.55}
$$

It can then be checked also that

$$S = \int \mathrm{d}^2x q_- q_+ \overline{q}_- \overline{q}_+ \left(K_0 - q_+ q_- K_1 \right) = \int \mathrm{d}^2x q_- q_+ \overline{q}_- \overline{q}_+ K_0. \tag{5.56}$$

Note that the K_1 term in Eq. (5.54) is indispensible and is entirely responsible for the K_1 terms in the expanded action, Eq. (5.55). However, in Eq. (5.56), the K_1 term is redundant and can be omitted, leading to a form for the action similar to that in the undeformed case, Eq. (5.20). The K_1 terms in Eq. (5.55) are generated from Eq. (5.56) by applying the relations of Eq. (5.52).

Finally, from Eq. (5.56), we see that, as in the undeformed case, the nilpotency of $q_\pm, \overline{q}_\pm$, which follows from that of $Q_\pm, \overline{Q}_\pm$ in Eq. (5.27), ensures

$$q_\pm S = \overline{q}_\pm S = 0. \tag{5.57}$$

5.4 One-Loop Corrections

The motivation for this work was to investigate the one-loop corrections for the deformed theory, and determine whether they could be interpreted in terms of a smearing of the background geometry as at the classical level. It seemed reasonable to do this order by order in $\frac{1}{M^2}$. It should be noted that K_i is a power series in $\frac{1}{M^2}$ starting at $\frac{1}{M^0}$ for even i and $\frac{1}{M}$ for odd i. As mentioned in §4.2, it is necessary to make a choice of method as the computation of the one-loop and higher quantum corrections for the undeformed Kähler σ-model may be performed in several different ways. In the undeformed case, the superspace calculation [77] is the most efficient and demonstrates the generally covariant form of the results: that is, that the action may be expressed in terms of the Kähler metric and its associated Riemann tensor. In the component formulation, the covariant form of the classical action is obtained upon integrating out the auxiliary fields, and computations up to four loops have also been carried out in this formalism [76]. In the non-anticommutative case, superspace computations have been performed in the four-dimensional context [86, 87] but the formalism is technically rather complex. On the other hand, integrating out the auxiliary fields in the deformed action, Eq. (5.55), would be difficult and, in any case, it is no longer clear if general covariance is a useful guide.

Accordingly, we decided to perform the calculation in the uneliminated component formulation. Unfortunately, it rapidly becomes apparent that there is a great number of diagrams to consider. As the diagrams with no factor of $\frac{1}{M}$ are to be found in the undeformed model, we have already seen that they may be expressed concisely in the form $q_-q_+\overline{q}_-^0\overline{q}_+^0\mathrm{tr}\ \ln\ K_{i\bar{j}}$. Similarly, it is possible to express those diagrams with a single factor of $\frac{1}{M^2}$ in the form $q_-q_+\mathcal{K}$ for some $\mathcal{K}$ which we refer to as the "kernel". Since $\overline{q}_-\overline{q}_+K_0 = \overline{F}^{\bar{i}}K_{0\bar{i}} - \overline{\psi}_+^{\bar{i}}\overline{\psi}_-^{\bar{j}}K_{0\bar{i}\bar{j}}$, with a similar expression for K_1, the action in Eq. (5.54), and hence the list of one-loop corrections, separates into four sections which can be independently written as q_-q_+ acting on a kernel. The kernel may thus be written

$$\begin{aligned}\mathcal{K}_B^{(1)} = {} & \frac{\partial^2 L_{M^2}}{\partial F^i \partial \overline{\phi}^{\bar{j}}} K^{i\bar{j}} - \frac{\partial^2 L_{M^2}}{\partial F^i \partial F^j} K^{i\bar{k}} K^{j\bar{l}} \left(K_{\bar{k}\bar{l}m} F^m - K_{\bar{k}\bar{l}mn} \psi^m \psi^n \right) \\ & + \frac{1}{24M^2} \left(A_1 + A_2 + A_3 + A_4 \right),\end{aligned} \tag{5.58}$$

where L_{M^2} is the M^2 term in the Lagrangian of Eq. (5.55) and A_{1-4} are expressed diagrammatically in Figs. 5.3-5.9.

In these diagrams a "propagator" in a loop denotes K^{-1} and vertices denote derivatives of K, while external lines attached to vertices represent the various fields according to the conventions in Fig. 5.1 and the convenient shorthand notations in Fig. 5.2.

Incoming arrows represent chiral fields whilst outgoing arrows represent antichiral fields. The ordering of fermion fields is fixed by the convention that we start at the left-most field at the top of the diagram and read clockwise around the loop. As an illustration of notation, the first diagram in A_1 in Fig. 5.3 represents

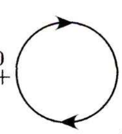
$$F^i F^j K_{ijk\bar{l}} K^{m\bar{l}} K^{k\bar{n}} \left(K_{m\bar{n}\bar{p}} \overline{F}^{\bar{p}} - K_{m\bar{n}\bar{p}\bar{q}} \overline{\psi}_+^{\bar{p}} \overline{\psi}_-^{\bar{q}} \right), \tag{5.59}$$

whilst the second represents

$$F^i \psi_+^j \psi_-^k K_{ip\bar{j}} K^{l\bar{j}} K_{jl\bar{k}} K^{m\bar{k}} \left(K_{m\bar{m}\bar{p}} \overline{F}^{\bar{p}} - K_{m\bar{m}\bar{p}\bar{q}} \overline{\psi}^{\bar{p}} \overline{\psi}^{\bar{q}} \right) K^{n\bar{m}} K_{kn\bar{n}} K^{p\bar{n}}, \tag{5.60}$$

where $K^{i\bar{j}} \equiv K_{i\bar{j}}^{-1}$.

Note that the undeformed corrections, as given in §5.2, correspond in this notation to

$$q_-q_+\overline{q}_-^0\overline{q}_+^0$$

, (5.61)

whilst the classical action corresponds to the operators acting on a single vertex.

Using $\partial_i K^{-1} = -K^{-1}\partial_i K K^{-1}$ the effect of $q_\pm$ is to add external lines and create new vertices. After acting on a diagram with q_-q_+, we obtain a set of diagrams which (unless they cancel with similar contributions from other kernel diagrams) correspond to viable one-loop Feynman diagrams, the vertex with the dot (or the blob) being the one from the deformed part of the action, and hence with an accompanying $\frac{1}{M^2}$ factor.

There are some intriguing patterns in the groups of diagrams appearing in A_{1-4}. For instance, one group of terms in A_1 is repeated in A_4 with the simple substitution of a blob for an incoming F; and another group of terms in A_1 may be obtained from the former group in A_1 by replacing a $\overline{\psi}_+$ followed by an adjacent $\overline{\psi}_-$ (or vice-versa) with an $\overline{F} - \overline{\psi}_+\overline{\psi}_-$. Finally, the graphs in A_3 are similar to those of A_2.

Having written the one-loop corrections in terms of the unbarred operators, it is tempting to wonder if the analogy with the undeformed case goes further so that we may write

$$S_B^{(1)} = \int d^2x q_- q_+ \overline{q}_- \overline{q}_+ \widetilde{\mathcal{K}}_B^{(1)} \tag{5.62}$$

for some underlying $\widetilde{\mathcal{K}}_B^{(1)}$, where $\overline{q}_\pm$ are the deformed operators constructed in Eq. (5.50); indeed this was our motivation for constructing these operators in the first instance. Unfortunately this turns out not to be the case, and this can be demonstrated as follows. Focusing on the set of graphs in $\mathcal{K}_B^{(1)}$ with five vertices, four with a single fermion and one with an F, it can be seen that the graphs with six vertices, five with one fermion and one with an F (and no derivatives) created by the action of $\overline{q}_+$ on this set do not cancel. In drawing this conclusion we can restrict attention to the effect of $\overline{q}_+^0$ since the remaining terms in $\overline{q}_+$ all contain derivatives. Since this is the only source of graphs of this type in $\overline{q}_+\mathcal{K}_B^{(1)}$, we see that $\overline{q}_+\mathcal{K}_B^{(1)} \neq 0$ (and, similarly, $\overline{q}_-\mathcal{K}_B^{(1)} \neq 0$). Therefore $q_-q_+\overline{q}\mathcal{K}_B^{(1)} \neq 0$ and $q_-q_+\overline{q}_-\mathcal{K}_B^{(1)} \neq 0$ (consider, for instance, those graphs for which q_-q_+ simply attaches an F at the vertex already containing an F) and so $\overline{q}_+S_B^{(1)} \neq 0$, $\overline{q}_-S_B^{(1)} \neq 0$. This immediately implies, due to the nilpotency of $\overline{q}_\pm$, that $S_B^{(1)}$ cannot be of the form Eq. (5.62). It is interesting that the classical behaviour is not reproduced at the quantum level, and in particular that the one-loop effective action is not invariant under $\overline{q}_\pm$ even though the classical action is. Having written down the one-loop divergences, we found that this model is in fact non-renormalisable. This is seen quite easily as we are able to insert multiple deformed vertices to produce one-loop divergences with arbitrarily large numbers of external states. We should clarify here that, although the undeformed theory is renormalisable for interaction terms of arbitrarily high order, a given Lagrangian will contain only a finite number of interaction terms. The non-renormalisability of the deformed model is due to the generation of arbitrarily high-order interaction terms from any interaction term of the classical action. A given action will thus require infinitely many parameters to absorb all divergences. Recall that we gave a general argument for the renormalisability of $\mathcal{N} = \frac{1}{2}$ theories in §4.3 which relied on non-Hermiticity. The fact that the present model is non-renormalisable does not invalidate that argument but it does require us to clarify the reason for its failure here. The ability of non-Hermiticity to "rescue" the renormalisability of non-anticommutative theories is reliant upon a "pseudo-R symmetry" of the models concerned which will be discussed in the following chapters. The non-renormalisability of the two-dimensional σ-model is due to the lack of such a symmetry.

5.5 Summary

We have constructed differential operators which encapsulate the supersymmetry in both the undeformed and deformed cases and which may be used to express the action of the non-linear σ-model. We have shown that these operators remain nilpotent in the deformed case and that, therefore, the classical action is invariant under the transformations induced by the operators. The one-loop effective action, however, we have shown not to be expressible in terms of the full set of operators in the deformed case despite being expressible in the form of the holomorphic operators acting on a kernel, which we have calculated to lowest order in the non-anticommutativity parameter. We have also shown that the arguments of §4.3 fail to apply in the two-dimensional case leading to the non-renormalisability of this model. The four-dimensional models we turn to in the remaining chapters, however, will prove to be renormalisable.

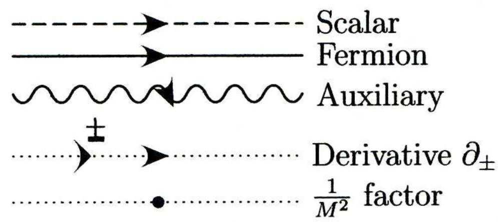


Figure 5.1: Conventions for Feynman diagrams in Figs. 5.3-5.9

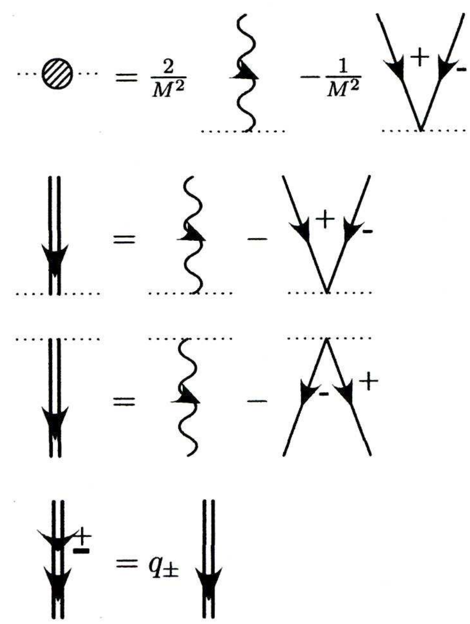


Figure 5.2: Shorthand notations used in Figs. 5.3-5.9

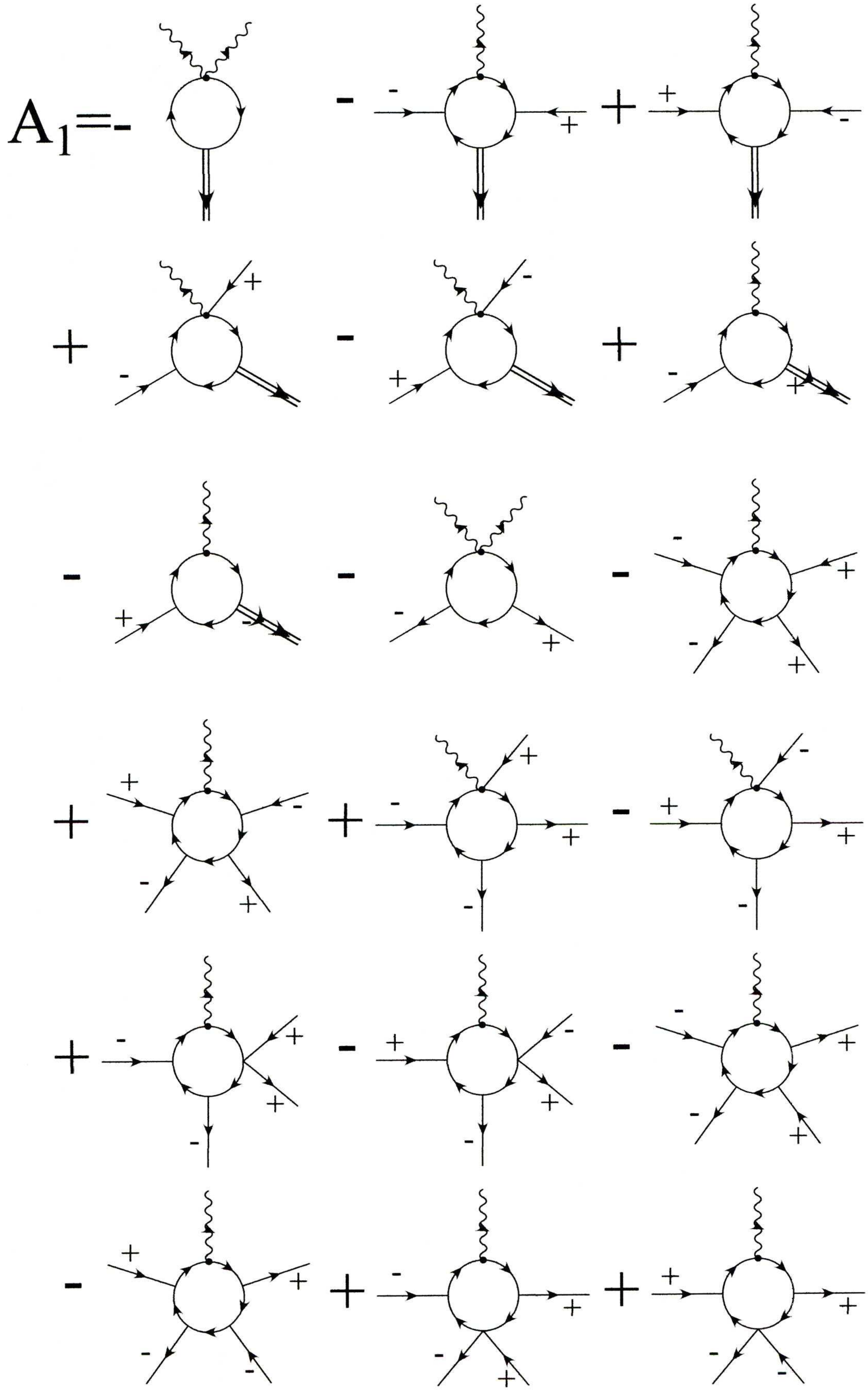


Figure 5.3: Diagrams for A_1

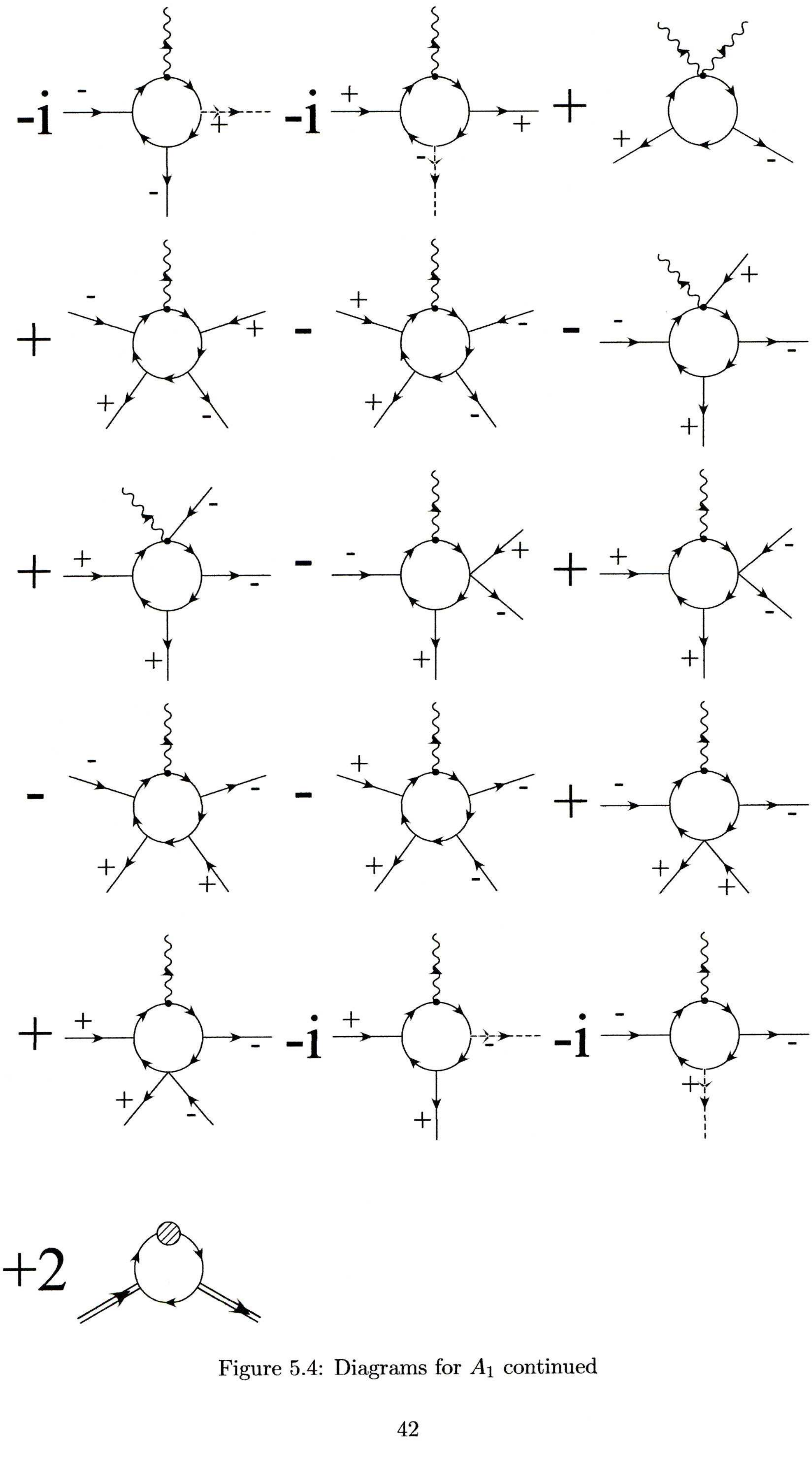


Figure 5.4: Diagrams for A_1 continued

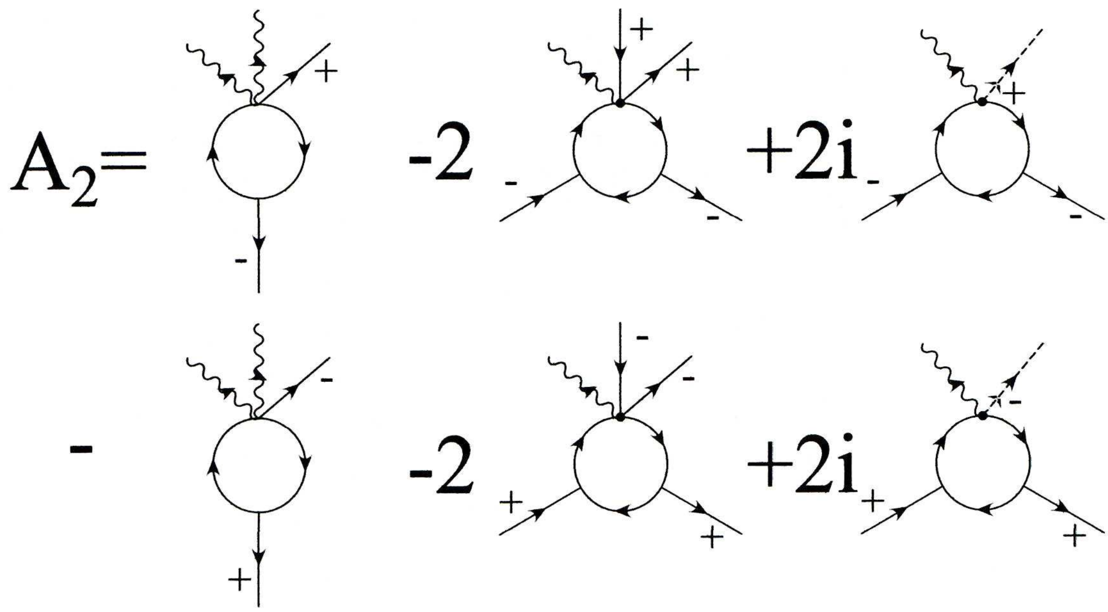


Figure 5.5: Diagrams for A_2

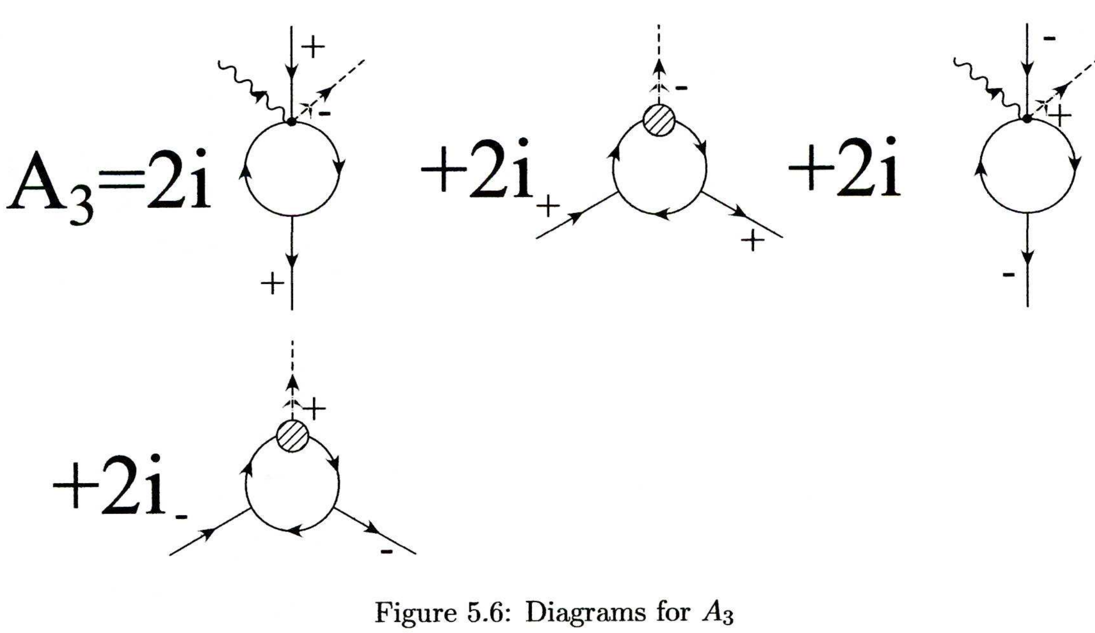


Figure 5.6: Diagrams for A_3

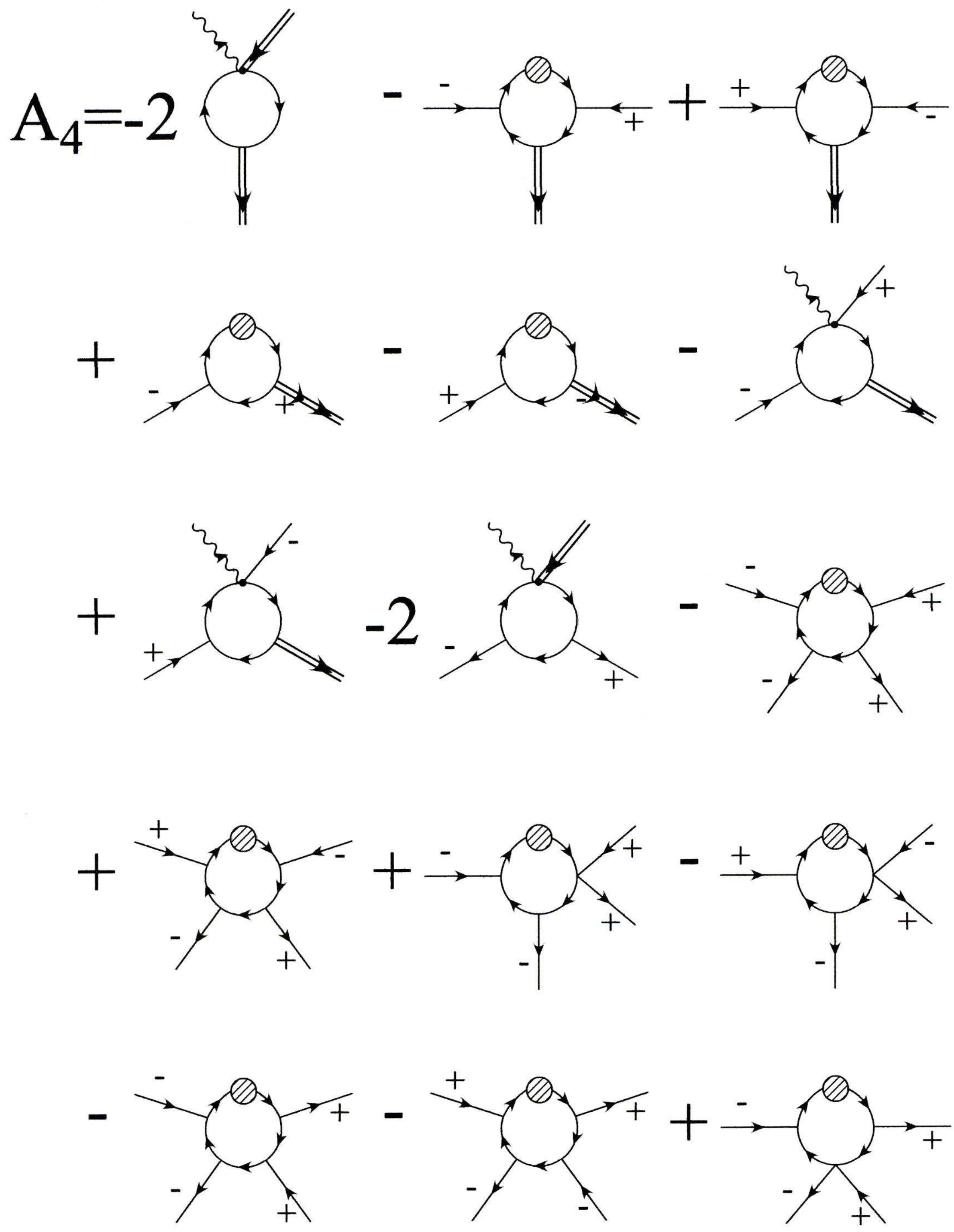


Figure 5.7: Diagrams for A_4

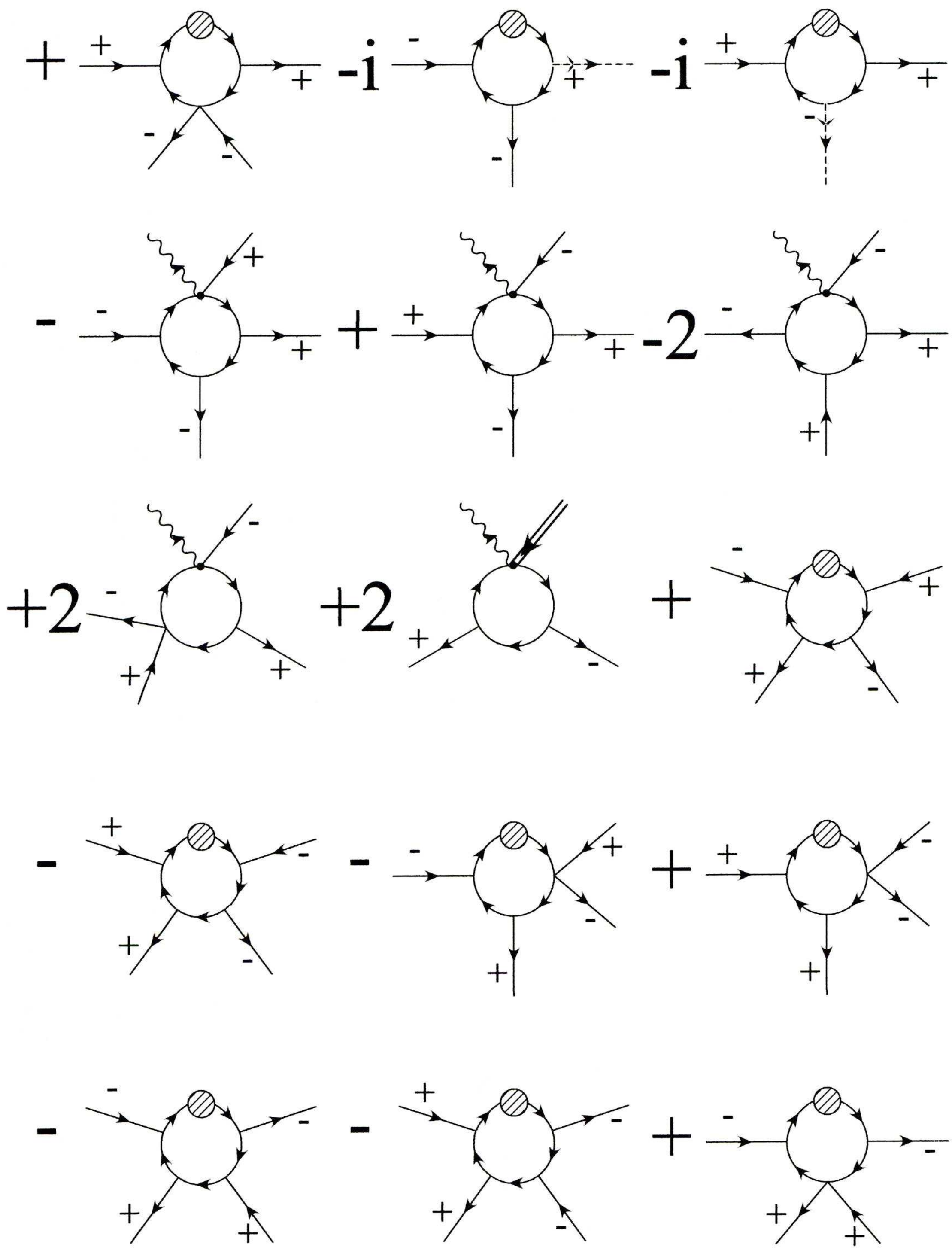


Figure 5.8: Diagrams for A_4 continued

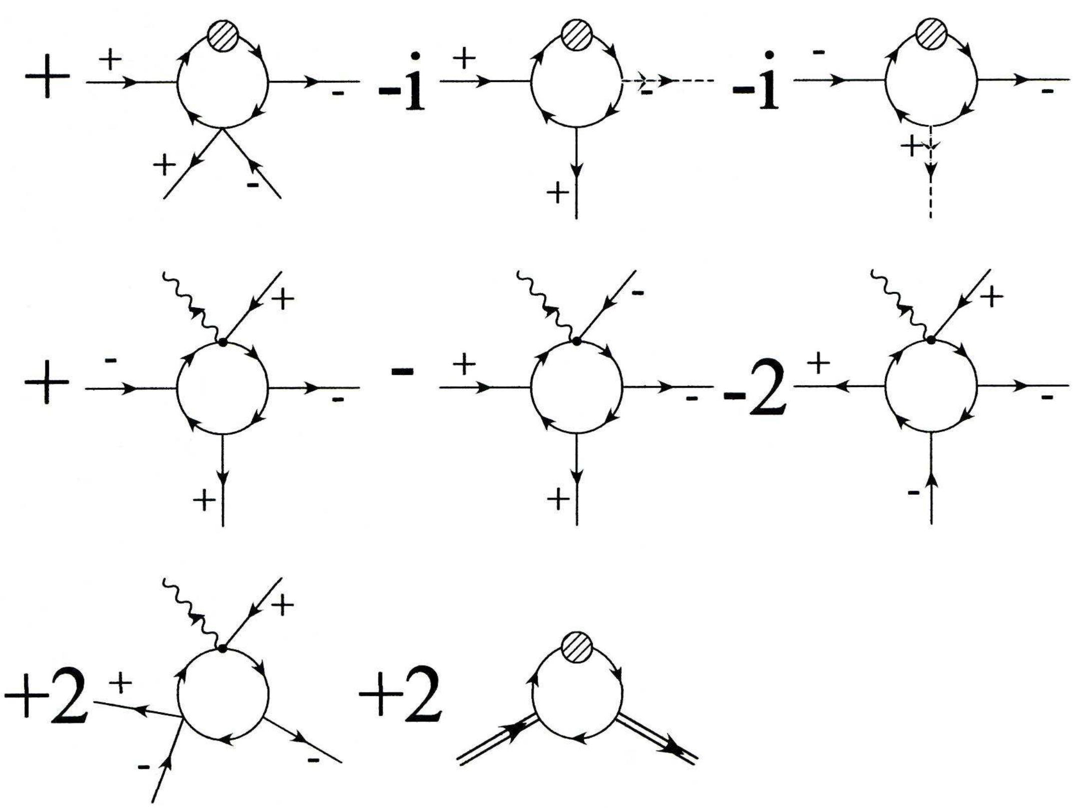


Figure 5.9: Diagrams for A_4 continued

Chapter 6

The Non-anticommutative Wess-Zumino Model

As the renormalisation of deformed gauged models has been studied in the past [50–53] and as such models meet with difficulties when introducing a superpotential (as will be discussed in chapter 8) in order to investigate the properties of a deformed superpotential, we focus in this chapter on a simple (ungauged) Wess-Zumino model [88]. The form of the classical action studied in this chapter is in accordance with the action given in Ref. [89]. In §§6.1,6.2, we derive differential operators representing the SUSY algebra in analogy to those derived for the σ-model in the previous chapter. The remainder of the chapter is dedicated to investigating the renormalisation of the model in both the uneliminated and eliminated formalisms and addressing the question of naturalness in the choice of certain parameters. We show that consistency between the two formalisms requires us to include separate couplings for all terms that may be generated by renormalisation of the classical action.

6.1 Representation of the Supersymmetry Algebra — the Undeformed Case

We follow the analysis of the previous chapter in determining differential operators which represent the supersymmetry algebra in the undeformed and deformed cases. As we are now working in four dimensions, we return to the standard notation for the supercharges:

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha}, \\ \overline{Q}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + 2i\theta^\alpha\partial_{\alpha\dot\alpha}, \end{aligned} \tag{6.1}$$

where

$$\partial_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} \tag{6.2}$$

and y^μ are the chiral coordinates as before.

The supercharges obey again the algebra

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0, \\ \left\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} &= 0, \\ \left\{\overline{Q}_{\dot\alpha}, Q_\alpha\right\} &= 2i\partial_{\alpha\dot\alpha}, \end{aligned} \tag{6.3}$$

and chiral and antichiral multiplets again take the form of Eq. (2.19).

Once again, we wish to represent the charges Q_α, $\overline{Q}_{\dot{\alpha}}$ by differential operators $q_\alpha, \overline{q}^0_{\dot{\alpha}}$ acting on the fields according to

$$\begin{aligned} [Q_\alpha, \Phi] &= q_\alpha \Phi, \\ \left[\overline{Q}_{\dot{\alpha}}, \Phi\right] &= \overline{q}^0_{\dot{\alpha}} \Phi \end{aligned} \tag{6.4}$$

and we find, in this case, that the differential operators take the form

$$\begin{aligned} \frac{1}{\sqrt{2}} q_\alpha &= \psi_\alpha \frac{\partial}{\partial \phi} + F \frac{\partial}{\partial \psi^\alpha} - i \left[\partial_{\alpha\dot{\alpha}} \overline{\phi}\right] \frac{\partial}{\partial \overline{\psi}_{\dot{\alpha}}} + i \left[\partial_{\alpha\dot{\alpha}} \overline{\psi}^{\dot{\alpha}}\right] \frac{\partial}{\partial \overline{F}}, \\ \frac{1}{\sqrt{2}} \overline{q}^0_{\dot{\alpha}} &= \overline{\psi}_{\dot{\alpha}} \frac{\partial}{\partial \overline{\phi}} - \overline{F} \frac{\partial}{\partial \overline{\psi}^{\dot{\alpha}}} - i \left[\partial_{\alpha\dot{\alpha}} \phi\right] \frac{\partial}{\partial \psi_\alpha} - i \left[\partial_{\alpha\dot{\alpha}} \psi^\alpha\right] \frac{\partial}{\partial F}, \end{aligned} \tag{6.5}$$

where again the superscript 0 is in anticipation of a different form for $\overline{q}_{\dot{\alpha}}$ in the deformed case.

As we did not need to deal with the Levi-Civita tensor in two dimensions, it is worth mentioning here that, whilst $\epsilon_{\alpha\beta}\psi^\beta = \psi_\beta$, for consistency we must define

$$\epsilon_{\alpha\beta} \frac{\partial}{\partial \psi_\beta} = -\frac{\partial}{\partial \psi^\alpha}. \tag{6.6}$$

6.2 Representation of the Supersymmetry Algebra — the Deformed Case

In this section we repeat the analysis of the previous section for the case of deformed supersymmetry. In this model we return to a general non-anticommutativity parameter

$$\begin{aligned} \left\{\overline{\theta}^{\dot{\alpha}}, \overline{\theta}^{\dot{\beta}}\right\} &= 0, \\ \left\{\theta^\alpha, \theta^\beta\right\} &= C^{\alpha\beta}, \end{aligned} \tag{6.7}$$

so that the supercharges again obey

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0, \\ \left\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} &= -4C^{\alpha\beta} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}}, \\ \left\{\overline{Q}_{\dot{\alpha}}, Q_\alpha\right\} &= 2i\partial_{\alpha\dot{\alpha}}, \end{aligned} \tag{6.8}$$

and the star-product is given by

$$\begin{aligned} \theta^\alpha \star \theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 + \frac{1}{2}C^{\alpha\beta}, \\ \theta^\alpha \star \theta^2 &= C^{\alpha\beta}\theta_\beta, \\ \theta^2 \star \theta^2 &= -\det C \equiv \frac{1}{M^2}. \end{aligned} \tag{6.9}$$

Note that, although we are working with a general non-anticommutativity parameter, we define a parameter M in terms of the determinant of $C^{\alpha\beta}$ in a manner analogous to that of the previous chapter. This allows us to retain the interpretation of M as the energy scale of the deformation.

We now wish to construct differential operators $\overline{q}_{\dot\alpha}$ representing the effects of $\overline{Q}_{\dot\alpha}$ in the deformed case in a similar manner to Eq. (6.4), extending $\overline{q}^0_{\dot\alpha}$ given in Eq. (6.5) for the undeformed case. We proceed as for the two-dimensional case by considering first the effects of $\overline{Q}_{\dot\alpha}$ on powers of Φ alone, since we will find it necessary again to deal with symmetrisation when we include products of both Φ and $\overline{\Phi}$. Using a slight modification of our definition of $I_r^{(n)}$ from the previous chapter,

$$I_r^{(n)}(\phi, F) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\xi \left(\frac{\xi}{2M}\right)^r \left(\phi + \frac{2\xi}{M}F\right)^n, \tag{6.10}$$

we find that

$$\Phi_\star^n = \left(1 + \theta q - \frac{1}{4}\theta^2 q^2\right)\left(I_0^{(n)} - q^2 I_1^{(n)}\right). \tag{6.11}$$

Acting on $\Phi_\star^n$, $\overline{Q}_{\dot\alpha}$ is found to be represented by

$$\overline{q}^{\Phi}_{\dot\alpha} = \overline{q}^0_{\dot\alpha} - i\,(qC\partial)_{\dot\alpha} + 4i\left(-q^2\left[\partial_{\alpha\dot\alpha}q^\alpha\right]\widetilde{\mathcal{O}} + \partial_{\alpha\dot\alpha}q^\alpha\mathcal{O} + \left[\partial_{\alpha\dot\alpha}q^\alpha\right]\mathcal{O}\right). \tag{6.12}$$

Here $\mathcal{O}$, $\widetilde{\mathcal{O}}$ are required to obey the same relations as in the two-dimensional case, Eq. (5.34), but their definitions must also be modified to account for the modification in the definition of $I_r^{(n)}$:

$$\begin{aligned}
\mathcal{O} &= \sum_{r=1}^{\infty} a_r \left(\frac{1}{4M^2}\right)^r \left(4F\frac{\partial}{\partial\phi}\right)^{2r-1}, \\
\widetilde{\mathcal{O}} &= \sum_{r=1}^{\infty} (2r-1)\, a_r \left(\frac{1}{4M^2}\right)^r \left(4F\frac{\partial}{\partial\phi}\right)^{2r-2},
\end{aligned} \tag{6.13}$$

where the a_r's are defined as in the previous chapter.

using the relations for $\mathcal{O}$, $\widetilde{\mathcal{O}}$ given in Eq. (5.34) along with

$$\overline{q}^0_{\dot\alpha} I_r^{(n)} = -i\left[\partial_{\alpha\dot\alpha}q^\alpha\right] I_{r+1}^{(n)}, \tag{6.14}$$

we can verify that the differential operators in Eq. (6.12) do indeed represent the operators $\overline{Q}_{\dot\alpha}$ according to

$$\left[\overline{Q}_{\dot\alpha}, \Phi_\star^n\right]_\star = \overline{q}^{\Phi}_{\dot\alpha}\Phi_\star^n. \tag{6.15}$$

Of course it is not sufficient to reproduce the effects of $\overline{Q}_{\dot\alpha}$ on star-products of Φ alone: we saw in the two-dimensional case that it was necessary to consider the effect on deformed versions of general polynomials in Φ and $\overline{\Phi}$ such as the Kähler potential. In the case of the four dimensional Wess-Zumino model, to investigate the divergence structure it would be sufficient to consider only the effects of $\overline{Q}_{\dot\alpha}$ on $\Phi \star \overline{\Phi}$ or on cubic superpotentials in either Φ or $\overline{\Phi}$, as higher powers would be non-renormalisable. However, in general, if one were interested in contributions to the effective action, one would need once again to consider deformed versions of general polynomials, and so we shall again take the Kähler potential as an example.

We find in this case that the (symmetrised) deformed Kähler potential may be expressed in terms of the differential operators as

$$\begin{aligned}
K_\star\left[\Phi, \overline{\Phi}\right] = &\left(1 + \theta q - \frac{1}{4}\theta^2 q^2\right)\left[1 + \overline{\theta}\overline{q}^{0\prime\prime} - \frac{1}{4}\overline{\theta}^2\left(\overline{q}^{0\prime\prime}\right)^2\right] \\
&\times\left[K_0\left(\phi, F, \overline{\phi}\right) - q^2 K_1\left(\phi, F, \overline{(\phi)}\right)\right] \\
&- \frac{1}{4M^2}\overline{\theta}^2\left(q'\right)^2\left(\partial''\right)^2 K_0\left(\phi, F, \overline{\phi}\right),
\end{aligned} \tag{6.16}$$

where

$$K_m\left(\phi,F,\overline{\phi}\right)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}\xi\left(\frac{\xi}{2M}\right)^m K\left(\phi+\frac{2\xi}{M}F,\overline{\phi}\right). \tag{6.17}$$

The version of the differential operator representing $\overline{Q}_{\dot{\alpha}}$ acting on the Kähler potential, $\overline{q}_{\dot{\alpha}}$ is then found to be given by

$$\begin{aligned}\overline{q}_{\dot{\alpha}}=\overline{q}^0_{\dot{\alpha}}-i\left(qC\partial\right)_{\dot{\alpha}}-\frac{i}{4M^2}\left(\partial''_{\alpha\dot{\alpha}}q''^{\alpha}q'^2+\partial'_{\alpha\dot{\alpha}}q'^{\alpha}q''^2\right)\\ +4i\left(-q'^2\left[\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\right]\widetilde{\mathcal{O}}+\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\mathcal{O}+\left[\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\right]\mathcal{O}\right).\end{aligned} \tag{6.18}$$

We can verify that these operators implement the operators $\overline{Q}_{\dot{\alpha}}$ according to

$$\left[\overline{Q}_{\dot{\alpha}},K_\star\right]_\star=\overline{q}_{\dot{\alpha}}K_\star, \tag{6.19}$$

using the properties of $\mathcal{O}$, $\widetilde{\mathcal{O}}$ (Eq. (5.52)) together with the analogue of Eq. (6.15):

$$\begin{aligned}\overline{q}'^0_{\dot{\alpha}}K_r&=-i\left[\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\right]K_{r+1},\\ q''_{\alpha}K_0&=q''_{\alpha}K_1=0.\end{aligned} \tag{6.20}$$

Furthermore, it is straightforward to check using

$$\left[\overline{q}^0_{\dot{\alpha}},F\frac{\partial}{\partial\phi}\right]=-i\left[\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\right] \tag{6.21}$$

which implies

$$\left[\overline{q}^0_{\dot{\alpha}},\mathcal{O}\right]=-i\left[\partial'_{\alpha\dot{\alpha}}q'^{\alpha}\right]\widetilde{\mathcal{O}}, \tag{6.22}$$

that the operators q_α in Eq. (6.5) and $\overline{q}_{\dot{\alpha}}$ in Eq. (6.18) satisfy the anticommutation relations of the SUSY algebra (Eq. (6.8)).

As in the two-dimensional case, the kinetic part of the standard Wess-Zumino model may be obtained from Eq. (6.16) by taking $K\left[\Phi,\overline{\Phi}\right]=\Phi\overline{\Phi}$:

$$\begin{aligned}S_{\text{kin}}&=\int\mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\Phi\star\overline{\Phi}=\frac{1}{16}\int\mathrm{d}^4xq^2\overline{q}^2\phi\overline{\phi}\\ &=\int\mathrm{d}^4x\left(\partial^\mu\overline{\phi}\partial_\mu\phi+i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi+F\overline{F}\right),\end{aligned} \tag{6.23}$$

so the kinetic terms are undeformed.

We see from Eq. (6.11) that the holomorphic potential terms are given by

$$\begin{aligned}S_W&=-\int\mathrm{d}^4x\int\mathrm{d}^2\theta\left[\frac{1}{2}m\Phi^2_\star+\frac{1}{6}y\Phi^3_\star\right]\\ &=\frac{1}{4}\int\mathrm{d}^4xq^2\left[\frac{1}{2}mI_0^{(2)}+\frac{1}{6}yI_0^{(3)}\right],\end{aligned} \tag{6.24}$$

which leads to

$$S_W=\int\mathrm{d}^4x\left[\frac{1}{2}m\left(\psi^2-F\phi\right)+\frac{1}{2}y\left(\phi\psi^2-F\phi^2\right)+\frac{1}{6}y\left(\det C\right)F^3\right]. \tag{6.25}$$

Since $\overline{\Phi}^n_\star=\overline{\Phi}^n$, the antiholomorphic potential terms are given by

$$\begin{aligned}S_{\overline{W}}&=\int\mathrm{d}^4x\int\mathrm{d}^2\overline{\theta}\left[\frac{1}{2}\overline{m}\overline{\Phi}^2+\frac{1}{6}\overline{y}\overline{\Phi}^3\right]\\ &=\frac{1}{4}\int\mathrm{d}^4x\overline{q}^2\left[\frac{1}{2}\overline{m}\overline{\phi}^2+\frac{1}{6}\overline{y}\overline{\phi}^3\right],\end{aligned} \tag{6.26}$$

which leads to

$$S_{\overline{W}} = \int \mathrm{d}^4x \left[\frac{1}{2}\overline{m}\left(\overline{\psi}^2 - \overline{F}\overline{\phi}\right) + \frac{1}{2}\overline{y}\left(\overline{\phi}\overline{\psi}^2 - \overline{F}\overline{\phi}^2\right)\right] \tag{6.27}$$

with no deformation as the deformed part of $\overline{q}$ in Eq. (6.18) has no effect on a function of $\overline{\phi}$. The full classical deformed action is therefore

$$\begin{aligned} S = \int \mathrm{d}^4x \Big(& \partial^\mu\overline{\phi}\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F - GF - \overline{G}\overline{F} \\ & + \frac{1}{2}y\phi\psi^2 + \frac{1}{2}\overline{y}\overline{\phi}\overline{\psi}^2 + \frac{1}{2}m\psi^2 + \frac{1}{2}\overline{m}\overline{\psi}^2 + \frac{1}{6}y\left(\det C\right)F^3\Big), \end{aligned} \tag{6.28}$$

where $G = m\phi + \frac{1}{2}y\phi^2$.

Note that the only additional term arising from the deformation is the $\det(C)F^3$ term; this is in agreement with the action given in §3.5 which was first derived in Ref. [37] by taking the standard undeformed action in superfields and replacing ordinary products by star-products.

In the undeformed case, expressions like those in Eqs. (6.23), (6.24), (6.26) in terms of q_α and $\overline{q}_{\dot\alpha}$ encapsulate the supersymmetry of the undeformed action S_0 due to the nilpotency of q_α, $\overline{q}_{\dot\alpha}$ and the fact that q_α, $\overline{q}_{\dot\alpha}$ annihilate functions of $\overline{\phi}$, ϕ respectively, leading to

$$q_\alpha S_0 = \overline{q}^0_{\dot\alpha} S_0 = 0. \tag{6.29}$$

In the deformed case, although

$$q_\alpha S_{\mathrm{kin}} = \overline{q}_{\dot\alpha} S_{\mathrm{kin}} = q_\alpha S_W = q_\alpha S_{\overline{W}} = \overline{q}_{\dot\alpha} S_{\overline{W}} = 0, \tag{6.30}$$

we find that

$$\overline{q}_{\dot\alpha} S_W \neq 0. \tag{6.31}$$

In fact, it is even found to be the case that

$$\overline{q}^0_{\dot\alpha} S_W \neq 0. \tag{6.32}$$

Thus it is only the transformations generated by Q_α that are a symmetry of the deformed action with potential. This is reminiscent of the fact that, in the two-dimensional case, where we considered only the kinetic part of the action derived from the Kähler potential, the quantum corrections were annihilated only by q_α, although in the two-dimensional case, the classical action was annihilated by both q_α and $\overline{q}_{\dot\alpha}$.

6.3 Renormalisation of the Deformed Wess-Zumino Model

In this section we discuss the renormalisation of the non-anticommutative Wess-Zumino model up to two loops. Two-loop calculations for the model were first performed in Ref. [89] whilst renormalisability in principle was demonstrated to all orders in Ref. [90]. We extend their calculation by including from the outset the full set of terms which can be generated by renormalisation.

The only effect of the non-anticommutativity in the component action of Eq. (6.28) is the final det C term. As mentioned above, the deformed action is only invariant under the transformations generated by Q_α. The deformed term is in fact separately invariant under these residual $\mathcal{N} = \frac{1}{2}$ transformations and so there is no reason for the coefficient to evolve in the same manner as the Yukawa coupling in the undeformed part of the action. Therefore, we are obliged to introduce this term with its own separate coefficient. This term is found to generate one-loop divergences whose cancellation requires $(\det C)F^2\overline{G}$ and $\overline{m}^2(\det C)F^2$ terms in the action, and these terms in turn generate further $(\det C)F^3$ divergences along with other new terms. All these terms should be included in the classical action with their own coefficients in order to guarantee renormalisability [91]. The additional terms which can be generated were demonstrated in Ref. [68]. In order to find the terms that can be generated, we must consider the following invariances of the action.

Firstly, the action has a "pseudo-R symmetry" under

$$\phi \to e^{-i\omega}\phi, \qquad F \to e^{i\omega}F, \qquad C^{\alpha\beta} \to e^{-2i\omega}C^{\alpha\beta}, \qquad y \to e^{i\omega}y, \tag{6.33}$$

where $\overline{F}$, $\overline{\phi}$, $\overline{y}$ transform with opposite charges to F, ϕ, y and ψ, $\overline{\psi}$ are neutral.

Secondly, the action also has a "pseudo-chiral symmetry" under

$$\phi \to e^{i\gamma}\phi, \qquad m \to e^{-2i\gamma}m, \qquad y \to e^{-3i\gamma}y, \tag{6.34}$$

with F and ψ transforming in a similar fashion to ϕ and barred quantities transforming with opposite charges. The divergent terms which can arise subject to these invariances consist (up to factors of $y\overline{y}$) only of

$$\begin{array}{lll} \overline{y}^{-1}(\det C)\,F^3, & \overline{y}^{-1}(\det C)\,F^2\overline{G}, & \overline{y}^{-1}(\det C)\,F\overline{G}^2, \\ \overline{y}^{-1}(\det C)\,\overline{G}^3, & \overline{y}^{-2}\overline{m}^2(\det C)\,F^2, & \overline{y}^{-2}\overline{m}^2(\det C)\,F\overline{G}, \\ \overline{y}^{-2}\overline{m}^2(\det C)\,\overline{G}^2, & \overline{y}^{-3}\overline{m}^4(\det C)\,F, & \overline{y}^{-3}\overline{m}^4(\det C)\,\overline{G}\ . \end{array} \tag{6.35}$$

It was suggested in Ref. [89] and proven in Ref. [92] that divergences form combinations of F and $\overline{G}$ and so only require a single coupling to remove divergences in terms with equal powers of F, for example $\det(C)F^2\overline{y}\overline{\phi}^2$ and $\det(C)F^2\overline{y}\overline{m}\overline{\phi}$. We have anticipated this above in our form for the divergences. The factors of $\overline{y}$ included in (6.35) are appropriate for invariance under the pseudo-chiral symmetry (Eq. (6.34)). These factors are not uniquely determined due to the fact that $y\overline{y}$ is invariant under this symmetry but our choices will be justified later. Each of these terms is separately $\mathcal{N}=\frac{1}{2}$-invariant and so there is no way to determine their coefficients from the classical action. However, we shall investigate whether renormalisabilty places any constraints on their values. Some of the terms in (6.35) could be omitted and still leave a renormalisable theory but we include all of these terms in the classical action for completeness. Note that, although Ref. [54] also mentions terms of the form $yF\psi C\psi$ and $\overline{\phi}^2\psi C\psi$, these terms are not possible in the ungauged case with only one chiral field due to the symmetry of $C^{\alpha\beta}$ and the antisymmetry of fermions. As such, the renormalisability of the model presented here would still apply in the non-supersymmetric bosonic case with an arbitrary symmetric coupling $C^{\mu\nu}$.

We are led, then, to the action

$$\begin{aligned} S = \int \mathrm{d}^4x \Big(& \partial^\mu\overline{\phi}\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F - GF - \overline{GF} \\ & + \frac{1}{2}y\phi\psi^2 + \frac{1}{2}\overline{y}\overline{\phi}\overline{\psi}^2 + \frac{1}{2}m\psi^2 + \frac{1}{2}\overline{m}\overline{\psi}^2 \\ & + \overline{y}^{-1}\left[\frac{1}{6}k_1F^3 + \frac{1}{2}k_2F^2\overline{G} + \frac{1}{2}k_3F\overline{G}^2 + \frac{1}{6}k_4\overline{G}^3\right] \\ & + \frac{1}{2}\overline{y}^{-2}\overline{m}^2\left[k_5F^2 + 2k_6F\overline{G} + k_7\overline{G}^2\right] + \overline{y}^{-3}\overline{m}^4\left[k_8F + k_9\overline{G}\right] \Big), \end{aligned} \tag{6.36}$$

where the coefficients k_{1-9} each contain a factor of (det C).

Note that we have no way to determine the renormalisation of (det C) independently, only that of the coefficients k_{1-9}.

6.3.1 Uneliminated One-Loop Renormalisation

We write the divergent contributions to the deformed part of the effective action in the form

$$\begin{aligned}\Gamma_C^{\text{pole}} = -\int d^4x \Big[& \overline{y}^{-1} \left(\overline{Z}_1 F^3 + \overline{Z}_2 F^2 \overline{G} + \overline{Z}_3 F \overline{G}^2 + \overline{Z}_4 \overline{G}^3\right) \\ & + \overline{y}^{-2} \overline{m}^2 \left(\overline{Z}_5 F^2 + \overline{Z}_6 F \overline{G} + \overline{Z}_7 \overline{G}^2\right) \\ & + \overline{y}^{-3} \overline{m}^4 \left(\overline{Z}_8 F + \overline{Z}_9 \overline{G}\right) \Big],\end{aligned} \tag{6.37}$$

where the overall minus sign is included simply to avoid more signs later.

Note that, as we work in the minimal subtraction scheme, $\overline{Z}_{1-9}$ contains no finite part. The divergent diagrams that contribute to $\overline{Z}_{1-9}$ can be divided into groups, each group containing diagrams which have the same internal lines and number of vertices as well as the same number of external auxiliary lines, thus differing only in the number of external scalar lines and corresponding $\overline{y}$ or $\overline{m}$ couplings. The divergent contributions within each group can be expressed purely in terms of F and $\overline{G}$. Using the Feynman diagram coventions shown in Fig. 6.1, Fig. 6.2 depicts, for each group at one loop, the example with the greatest number of external $\overline{\phi}$ lines. Their divergent contributions are given, diagram by diagram, in Table 6.1 and given in total by

$$\begin{aligned}\overline{Z}_1^{(1)} &= \frac{1}{2} k_2 y \overline{y} L, \\ \overline{Z}_2^{(1)} &= (2k_1 + 4k_2 + 3k_3)\, y\overline{y} L, \\ \overline{Z}_3^{(1)} &= \left(2k_2 + 4k_3 + \frac{5}{2} k_4\right) y\overline{y} L, \\ \overline{Z}_4^{(1)} &= 0, \\ \overline{Z}_5^{(1)} &= (k_1 + 2k_2 + k_3 + k_6)\, y\overline{y} L, \\ \overline{Z}_6^{(1)} &= (k_2 + 2k_3 + k_4 + 2k_5 + 4k_6 + 3k_7)\, y\overline{y} L, \\ \overline{Z}_7^{(1)} &= 0, \\ \overline{Z}_8^{(1)} &= (k_5 + 2k_6 + k_7 + k_9)\, y\overline{y} L, \\ \overline{Z}_9^{(1)} &= 0,\end{aligned} \tag{6.38}$$

where

$$L = \frac{1}{16\pi^2 \epsilon}. \tag{6.39}$$

Factors of $y\overline{y}L$ are suppressed in Table 6.1.

These divergences are cancelled by replacing the parameters y, $\overline{y}$, k_{1-9} and fields ϕ, $\overline{\phi}$, F, $\overline{F}$, ψ, $\overline{\psi}$ by corresponding appropriately chosen bare quantities y_B, $\overline{y}_B$, k_{1B-9B}, ϕ_B, $\overline{\phi}_B$, F_B, $\overline{F}_B$, ψ_B, $\overline{\psi}_B$ where the bare scalar and fermion fields are given by, for instance, $\phi_B = Z^{\frac{1}{2}}\phi$ with the same Z for each bare field in the case of the Wess-Zumino model. However, the auxiliary fields do not necessarily renormalise in such a simple way. Although non-linear auxiliary-field renormalisation, such as was discussed in §4.3,

	$\overline{Z}_1$	$\overline{Z}_2$	$\overline{Z}_3$	$\overline{Z}_5$	$\overline{Z}_6$	$\overline{Z}_8$
a	$\frac{1}{2}k_2$					
b		$4k_2$		$2k_2$		
c		$2k_1$		k_1		
d		$3k_3$		k_3+k_6		
e			$4k_3$		$2(k_3+2k_6)$	$2k_6$
f			$2k_2$		k_2+2k_5	k_5
g			$\frac{5}{2}k_4$		k_4+3k_7	k_7+k_9

Table 6.1: Divergent contributions from Fig. 6.2

is not obligatory in the case of the Wess-Zumino model as it is for gauged theories, we nevertheless explore the freedom of making such renormalisations which will introduce an element of arbitrariness into the β-functions for k_i. Specifically, we set

$$\begin{aligned} F_B &= Z^{\frac{1}{2}}F, \\ \overline{F}_B &= Z^{\frac{1}{2}}\overline{F} + \overline{y}_B^{-1}\left(\frac{1}{2}R_B F_B^2 + S_B F_B \overline{G}_B + \frac{1}{2}T_B\overline{G}_B^2\right) \\ &\quad + \overline{m}_B^2\left(U_B F_B + V_B\overline{G}_B\right) + \overline{y}_B\overline{m}_B^4 W_B, \end{aligned} \tag{6.40}$$

where R_B, S_B, T_B, U_B, V_B and W_B contain divergent contributions only.

The non-renormalisation theorem leads to

$$\begin{aligned} y_B &= \mu^{\frac{1}{2}\epsilon}Z^{-\frac{3}{2}}y, \\ \overline{y}_B &= \mu^{\frac{1}{2}\epsilon}Z^{-\frac{3}{2}}\overline{y}, \\ m_B &= Z^{-1}m, \\ \overline{m}_B &= Z^{-1}\overline{m}, \end{aligned} \tag{6.41}$$

where μ is the usual dimensional-regularisation mass parameter, and hence

$$\overline{G}_B = Z^{-\frac{1}{2}}\left(\mu^{\frac{1}{2}\epsilon}\overline{y}\overline{\phi}^2 + \overline{m}\overline{\phi}\right). \tag{6.42}$$

We write

$$\begin{aligned} k_{iB} &= k_i + \sum_n k_{iB}^{(n)}, \qquad i = 1\ldots 4, \\ k_{iB} &= \mu^{\epsilon}\left(k_i + \sum_n k_{iB}^{(n)}\right), \qquad i = 5,6,7, \\ k_{iB} &= \mu^{-2\epsilon}\left(k_i + \sum_n k_{iB}^{(n)}\right), \qquad i = 8,9, \end{aligned} \tag{6.43}$$

where n denotes the loop order, and noting that $(\det C)$ has dimension 2, and

$$k_{iB}^{(n)} = \sum_{m=1}^{n}\frac{\kappa_i^{(n,m)}}{\epsilon^m}, \qquad i = 1\ldots 9, \tag{6.44}$$

where the first index on $\kappa_i^{(n,m)}$ denotes the loop order and the second denotes the order of the pole in ϵ.

We find, from the bare version of Eq. (6.28), that

$$
\begin{aligned}
k_{1B}^{(1)} &= 6\overline{Z}_1^{(1)} - 3Z^{(1)}k_1 - 3R_B^{(1)}, \\
k_{2B}^{(1)} &= 2\overline{Z}_2^{(1)} - 2Z^{(1)}k_2 + \left(R_B^{(1)} - 2S_B^{(1)}\right), \\
k_{3B}^{(1)} &= 2\overline{Z}_3^{(1)} - Z^{(1)}k_3 + \left(2S_B^{(1)} - T_B^{(1)}\right), \\
k_{4B}^{(1)} &= 6\overline{Z}_4^{(1)} + 3T_B^{(1)}, \\
k_{5B}^{(1)} &= 2\overline{Z}_5^{(1)} - 2Z^{(1)}k_5 - 2U_B^{(1)}, \\
k_{6B}^{(1)} &= \overline{Z}_6^{(1)} - Z^{(1)}k_6 + \left(U_B^{(1)} - V_B^{(1)}\right), \\
k_{7B}^{(1)} &= 2\overline{Z}_7^{(1)} + 2V_B^{(1)}, \\
k_{8B}^{(1)} &= \overline{Z}_8^{(1)} - Z^{(1)}k_8 - W_B^{(1)}, \\
k_{9B}^{(1)} &= \overline{Z}_9^{(1)} + W_B^{(1)}.
\end{aligned} \tag{6.45}
$$

R_B, S_B, T_B have similar expansions to k_{1B-4B} in Eqs. (6.43), (6.44) but with, for example,

$$
R_B = \sum_{m=1}^{n} \frac{r^{(n,m)}}{\epsilon^m}, \tag{6.46}
$$

whilst U_B, V_B have similar expansions to k_{5B-7B} and W_B to k_{8B-9B}

Writing $\beta_i = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} k_i$ and requiring that k_{iB} in Eq. (6.43) be independent of μ, we then find that

$$
\beta_i^{(1)} = \kappa_i^{(1,1)}, \qquad i = 1 \ldots 9. \tag{6.47}
$$

The β-functions for y, $\overline{y}$ are defined similarly; we have

$$
Z^{(1)} = -\frac{y\overline{y}}{16\pi^2\epsilon}, \tag{6.48}
$$

and then, by Eq. (6.41),

$$
\beta_y^{(1)} = \frac{y\overline{y}}{16\pi^2} \times \frac{3}{2} y, \tag{6.49}
$$

with a similar expression for $\beta_{\overline{y}}^{(1)}$.

Writing $r^{(1,1)} = \frac{y\overline{y}}{16\pi^2} r_1$ etc., we have from Eqs. (6.38), (6.43)-(6.48)

$$
\begin{aligned}
\beta_1^{(1)} &= \kappa_1^{(1,1)} = \frac{y\overline{y}}{16\pi^2} 3\left(k_1 + k_2 - r_1\right), \\
\beta_2^{(1)} &= \kappa_2^{(1,1)} = \frac{y\overline{y}}{16\pi^2} \left(4k_1 + 10k_2 + 6k_3 + r_1 - 2s_1\right), \\
\beta_3^{(1)} &= \kappa_3^{(1,1)} = \frac{y\overline{y}}{16\pi^2} \left(4k_2 + 9k_3 + 5k_4 + 2s_1 - t_1\right), \\
\beta_4^{(1)} &= \kappa_4^{(1,1)} = \frac{y\overline{y}}{16\pi^2} 3t_1, \\
\beta_5^{(1)} &= \kappa_5^{(1,1)} = \frac{y\overline{y}}{16\pi^2} 2\left(k_1 + 2k_2 + k_3 + k_5 + k_6 - u_1\right), \\
\beta_6^{(1)} &= \kappa_6^{(1,1)} = \frac{y\overline{y}}{16\pi^2} \left(k_2 + 2k_3 + k_4 + 2k_5 + 5k_6 + 3k_7 + u_1 - v_1\right), \\
\beta_7^{(1)} &= \kappa_7^{(1,1)} = \frac{y\overline{y}}{16\pi^2} 2v_1,
\end{aligned} \tag{6.50}
$$

$$\beta_8^{(1)} = \kappa_8^{(1,1)} = \frac{y\overline{y}}{16\pi^2}\left(k_5 + 2k_6 + k_7 + k_8 + k_9 - w_1\right),$$
$$\beta_9^{(1)} = \kappa_9^{(1,1)} = \frac{y\overline{y}}{16\pi^2} w_1.$$

The results adopt their simplest from for $r_1 = s_1 = t_1 = u_1 = v_1 = w_1 = 0$, in which case the β-functions for k_4, k_7 and k_9 are identically zero and, therefore, the corresponding terms could be omitted from the action. This feature will persist at two loops, although only in the present case: β_4, β_7, β_9 would all acquire contributions in the case of more than one chiral multiplet if the $yF\psi C\psi$ and $\overline{\phi}^2\psi C\psi$ interactions were included. In this simple case, we also see that once we have a non-zero k_1, we inevitably generate k_2 and k_5, and from these we generate k_3, k_6 and k_8. This is why it was, in principle, necessary to give the $\overline{y}^{-1}F^3$ term its own coupling, k_1, rather than $y\overline{y}$, since in general $\beta_1 \neq \mu\frac{\mathrm{d}}{\mathrm{d}\mu}y\overline{y}$, and why we had to introduce the other terms corresponding to k_{2-9} also. In fact, by taking

$$r_1 = k_2, \qquad s_1 = t_1 = u_1 = v_1 = w_1 = 0, \tag{6.51}$$

we can make $\beta_1^{(1)}$ consistent with $k_1 = y\overline{y}$ corresponding to a coefficient of y for the F^3 term as in the classical action, Eq. (6.28), but we cannot use the freedom in choosing r_1, s_1, t_1, u_1, v_1, w_1 to maintain zero values for k_{2-9}.

6.3.2 Eliminated One-Loop Renormalisation

The equations of motion for the auxiliary fields are given by

$$\begin{aligned} F &= \overline{G}, \\ \overline{F} &= G + \overline{y}^{-1}\left[\frac{1}{2}k_1F^2 + k_2F\overline{G} + \frac{1}{2}k_3\overline{G}^2\right] + \overline{y}^{-2}\overline{m}^2\left[k_5F + k_6\overline{G}\right] + \overline{y}^{-3}\overline{m}^4k_8, \end{aligned} \tag{6.52}$$

and using these to eliminate the auxiliary fields from the action gives

$$\begin{aligned} S = \int \mathrm{d}^4x \Big\{ & \partial^\mu\overline{\phi}\partial_\mu\phi + i\overline{\psi}\overline{\sigma}\partial_\mu\psi \\ & - G\overline{G} + \frac{1}{2}y\phi\psi^2 + \frac{1}{2}\overline{y}\overline{\phi}\overline{\psi}^2 + \frac{1}{2}m\psi^2 + \frac{1}{2}\overline{m}\overline{\psi}^2 \\ & + \frac{1}{6}\lambda_1\overline{y}^{-1}\overline{G}^3 + \frac{1}{2}\overline{m}^2\overline{y}^{-2}\lambda_2\overline{G}^2 + \overline{m}^4\overline{y}^{-3}\lambda_3\overline{G}\Big\}, \end{aligned} \tag{6.53}$$

where

$$\begin{aligned} \lambda_1 &= k_1 + 3(k_2 + k_3) + k_4, \\ \lambda_2 &= k_5 + 2k_6 + k_7, \\ \lambda_3 &= k_8 + k_9. \end{aligned} \tag{6.54}$$

Writing the divergent contributions to the deformed part of the effective action in the eliminated case as

$$\Gamma^{\text{pole}}_{C(\text{elim})} = -\int \mathrm{d}^4x \left[Y_1\overline{y}^{-1}\overline{G}^3 + \overline{y}^{-2}\overline{m}^2Y_2\overline{G}^2 + \overline{y}^{-3}\overline{m}^4Y_3\overline{G}\right] \tag{6.55}$$

(introducing an overall negative sign as in Eq. (6.55)), we have

$$\begin{aligned} \lambda_{1B} &= \lambda_1 + 6Y_1, \\ \lambda_{2B} &= \lambda_2 + 2Y_2, \\ \lambda_{3B} &= \lambda_3 + Y_3. \end{aligned} \tag{6.56}$$

We find from the eliminated diagrams (an example of which is depicted in Fig. 6.2(h)) that

$$\begin{aligned} Y_1^{(1)} &= \frac{5}{2}\lambda_1 y\overline{y}L, \\ Y_2^{(1)} &= (\lambda_1 + 3\lambda_2)\, y\overline{y}L, \\ Y_3^{(1)} &= (\lambda_2 + \lambda_3)\, y\overline{y}L. \end{aligned} \tag{6.57}$$

Then writing

$$\begin{aligned} \lambda_{1B} &= \lambda_1 + \sum_n \lambda_{1B}^{(n)}, \\ \lambda_{2B} &= \mu^{-\epsilon}\left(\lambda_2 + \sum_n \lambda_{2B}^{(n)}\right), \\ \lambda_{3B} &= \mu^{-2\epsilon}\left(\lambda_3 + \sum_n \lambda_{3B}^{(n)}\right), \end{aligned} \tag{6.58}$$

where

$$\lambda_{iB}^{(n)} = \sum_{m=1}^{n} \frac{L_i^{(n,m)}}{\epsilon^m}, \qquad i = 1\ldots 3, \tag{6.59}$$

we find, from Eqs. (6.41), (6.42), (6.53), (6.56), (6.57), that

$$\begin{aligned} L_1^{(1,1)} &= \frac{1}{16\pi^2} 15\lambda_1, \\ L_2^{(1,1)} &= \frac{1}{16\pi^2}\left(2\lambda_1 + 6\lambda_2\right), \\ L_3^{(1,1)} &= \frac{1}{16\pi^2}\left(\lambda_2 + \lambda_3\right) \end{aligned} \tag{6.60}$$

and

$$\beta_{\lambda_i}^{(1)} = L_i^{(1,1)}. \tag{6.61}$$

Substituting in the values for k_{1-9} from the uneliminated case, we find

$$\begin{aligned} \lambda_{1B} &= k_{1B} + k_{4B} + 3\left(k_{2B} + k_{3B}\right), \\ \lambda_{2B} &= k_{5B} + 2k_{6B} + k_{7B}, \\ \lambda_{3B} &= k_{8B} + k_{9B}, \end{aligned} \tag{6.62}$$

as should be the case for consistency between the two formalisms. Using Eqs. (6.43), (6.44), (6.50), (6.58), (6.59), (6.60), we find that this is satisfied at one loop regardless of the values of R_B, S_B, T_B, U_B, V_B, W_B in Eq. (6.40).

6.3.3 Naturalness in the Action

The original deformed Wess-Zumino action of Eq. (6.28) corresponded to the values $k_1 = y\overline{y}$ and $k_{2-9} = 0$. However, as was emphasised earlier, the more general action of Eq. (6.36) is invariant under $\mathcal{N} = \frac{1}{2}$ transformations regardless of the values of k_{1-9} and we saw from Eq. (6.50) that the choice $k_1 = y\overline{y}$, k_{2-9} is not maintained by renormalisation. We wish to investigate whether there is a choice of values for k_{1-9}

which *is* preserved by renormalisation and which would be, in some sense, natural. More precisely, we wish to write

$$\begin{aligned}
k_i &= a_i\,(y\overline{y})^{\rho}\,, \qquad & i &= 1,\ldots,4,\\
k_i &= a_i\,(y\overline{y})^{\sigma}\,, \qquad & i &= 5,6,7,\\
k_i &= a_i\,(y\overline{y})^{\tau}\,, \qquad & i &= 8,9,
\end{aligned} \tag{6.63}$$

where $a_i, i = 1\ldots 9$ are simply scale-independent numbers (independent of y or $\overline{y}$). This entails

$$\begin{aligned}
\frac{\beta_1^{(1)}}{k_1} = \frac{\beta_2^{(1)}}{k_2} = \frac{\beta_3^{(1)}}{k_3} = \frac{\beta_4^{(1)}}{k_4} &= \rho\left(\frac{\beta_y^{(1)}}{y} + \frac{\beta_{\overline{y}}^{(1)}}{\overline{y}}\right),\\
\frac{\beta_5^{(1)}}{k_5} = \frac{\beta_6^{(1)}}{k_6} = \frac{\beta_7^{(1)}}{k_7} &= \sigma\left(\frac{\beta_y^{(1)}}{y} + \frac{\beta_{\overline{y}}^{(1)}}{\overline{y}}\right),\\
\frac{\beta_8^{(1)}}{k_8} = \frac{\beta_9^{(1)}}{k_9} &= \tau\left(\frac{\beta_y^{(1)}}{y} + \frac{\beta_{\overline{y}}^{(1)}}{\overline{y}}\right).
\end{aligned} \tag{6.64}$$

Using Eqs. (6.49), (6.50), we obtain the following system of equations:

$$\begin{aligned}
(3-3\rho)\,k_1 + 3k_2 - 3r_1 &= 0,\\
4k_1 + (10-3\rho)\,k_2 + 6k_3 + r_1 - 2s_1 &= 0,\\
4k_2 + (9-3\rho)\,k_3 + 5k_4 + 2s_1 - t_1 &= 0,\\
-3\rho k_4 + 3t_1 &= 0,\\
2k_1 + 4k_2 + 2k_3 + (2-3\sigma)\,k_5 + 2k_6 - 2u_1 &= 0,\\
k_2 + 2k_3 + k_4 + 2k_5 + (5-3\sigma)\,k_6 + 3k_7 + u_1 - v_1 &= 0,\\
-3\sigma k_7 + 2v_1 &= 0,\\
k_5 + 2k_6 + k_7 + (1-3\tau)\,k_8 + k_9 - w_1 &= 0,\\
-3\tau k_9 + w_1 &= 0.
\end{aligned} \tag{6.65}$$

We can solve, for instance, the first three of these equations successively for r_1 then s_1 then t_1. The fourth equation then gives the constraint

$$(15-3\rho)\,\lambda_1 = 0. \tag{6.66}$$

Dealing with the fifth to seventh equations similarly, we find

$$2\lambda_1 + (6-3\sigma)\,\lambda_2 = 0, \tag{6.67}$$

whilst the eighth and ninth lead to

$$\lambda_2 + (1-3\tau)\,\lambda_3 = 0. \tag{6.68}$$

These conditions are equivalent to the equations that would arise from Eqs. (6.60), (6.61) in the eliminated case had we sought such a Renormalisation Group-invariant solution. Therefore, as we should expect, the eliminated and uneliminated forms of the theory contain the same information.

If, say, $\lambda_1 \neq 0$, implying $\lambda_{2,3} \neq 0$ by Eqs. (6.67), (6.68), then we require $\rho = 5$ but then we can choose r_1, s_1, t_1, u_1, v_1, w_1 to satisfy Eqs. (6.65) for any k_i and there are similar solutions with $\lambda_1 = 0$, $\lambda_{2,3} \neq 0$, $\sigma = 2$ and $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, $\tau = \frac{1}{3}$, all of which have arbitrary k_i. In the case

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \tag{6.69}$$

where the deformed potential vanishes in the eliminated case, there is, in general, no constraint on ρ, σ, τ. In this case, for any values of k_i satisfying Eq. (6.69) and any ρ, σ, τ, we can again choose r_1, s_1, t_1, u_1, v_1, w_1 to satisfy Eqs. (6.65).

There are four special cases, however, that place constraints on k_i in order to satisfy two extra conditions:

$$\begin{aligned} r_1 = s_1 = t_1 = u_1 = v_1 = w_1 = 0, \\ \rho = \sigma = \tau, \end{aligned} \tag{6.70}$$

which seem to us to be rather natural constraints, the first removing non-linear renormalisations from the auxiliary fields which were included in this model merely for completeness, and the second insisting that all deformed couplings scale in the same fashion.

The four special cases are: firstly,

$$\begin{aligned} &k_1 = \frac{1}{4}k_2 = \frac{3}{8}k_3 = \frac{1}{2}k_5 = \frac{3}{4}k_6 = 3k_8, \\ &k_4 = k_9 = 0, \\ &\rho = \sigma = \tau = 5, \end{aligned} \tag{6.71}$$

secondly,

$$\begin{aligned} &k_1 = k_2 = -\frac{3}{4}k_3, \quad k_6 = -\frac{5}{3}k_1 + 2k_5, \\ &k_8 = -\frac{2}{3}k_1 + k_5, \quad k_4 = k_7 = k_9 = 0, \\ &\rho = \sigma = \tau = 2, \end{aligned} \tag{6.72}$$

thirdly

$$\begin{aligned} &k_1 = -\frac{3}{2}k_2 = 3k_3, \quad k_5 = -2k_6, \\ &k_4 = k_7 = k_9 = 0, \quad \rho = \sigma = \tau = \frac{1}{3}, \end{aligned} \tag{6.73}$$

and fourthly

$$\begin{aligned} &k_1 = -k_2 = k_3 = -k_4, \quad k_5 = -k_6 = k_7, \\ &k_8 = -k_9, \qquad\qquad \rho = \sigma = \tau = 0. \end{aligned} \tag{6.74}$$

Of these cases, the third and fourth are particularly interesting since they will also prove to be valid in a similar way (with no non-linear renormalisation of $\overline{F}$) at two loops. This may be related to the fact that Eqs. (6.73), (6.74) correspond to actions of the simple forms

$$\begin{aligned} S = \int \mathrm{d}^4x \Big\{ &\partial^\mu\overline{\phi}\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F - GF - \overline{GF} \\ &+ \frac{1}{2}y\phi\psi^2 + \frac{1}{2}\overline{y}\overline{\phi}\overline{\psi}^2 + \frac{1}{2}m\psi^2 + \frac{1}{2}\overline{m}\overline{\psi}^2 \\ &+ F\left[\frac{1}{6}k_1\overline{y}^{-1}\left(F - \overline{G}\right)^2 + \frac{1}{2}k_5\overline{y}^{-2}\overline{m}^2\left(F - \overline{G}\right) + k_8\overline{y}^{-3}\overline{m}^4\right]\Big\} \end{aligned} \tag{6.75}$$

	$\overline{Z}_1$
a	$\frac{1}{2}Ik_2$
b	$2k_2$
c	k_1
d	$\frac{3}{2}k_3$

Table 6.2: Divergent contributions from Fig. 6.3

and

$$\begin{aligned} S = \int \mathrm{d}^4x \Big\{ & \partial^\mu\overline{\phi}\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F - GF - \overline{GF} \\ & + \frac{1}{2}y\phi\psi^2 + \frac{1}{2}\overline{y\phi\psi}^2 + \frac{1}{2}m\psi^2 + \frac{1}{2}\overline{m\psi}^2 \\ & + \frac{1}{6}k_1\overline{y}^{-1}\left(F-\overline{G}\right)^3 + \frac{1}{2}k_5\overline{y}^{-2}\overline{m}^2\left(F-\overline{G}\right)^2 \\ & + k_8\overline{y}^{-3}\overline{m}^4\left(F-\overline{G}\right)\Big\} \end{aligned} \tag{6.76}$$

respectively. The equations of motion for F and $\overline{F}$ in Eq. (6.52) are then particularly simple upon applying Eqs. (6.73), (6.74). In particular, the equation of motion for F becomes linear in $\overline{G}$ upon applying the equation for $\overline{F}$, $F = \overline{G}$. We regard (6.73) and (6.74), therefore, as natural forms for the action. It is also rather interesting that the values $\rho = 2$ and $\rho = \frac{1}{3}$ have a significance even in the massless case, as we see in Eqs. (6.72), (6.73), despite these values originally arising for the coefficients of the massive terms in Eqs. (6.67), (6.68).

6.3.4 Uneliminated Two-Loop Renormalisation

We now turn to the two-loop calculation which, as we shall see, follows a very similar pattern to the one-loop calculation. At two loops we have

$$Z^{(2)} = -I\frac{(y\overline{y}L)^2}{\epsilon^2}, \tag{6.77}$$

where

$$I = 1 - \frac{1}{2}\epsilon \tag{6.78}$$

which, by Eq. (6.41), leads to

$$\beta_y^{(2)} = -\frac{3}{2}y\overline{y}L^2y. \tag{6.79}$$

Examples of each group of divergent two-loop diagrams, except for those contributing to $\overline{Z}_4$, $\overline{Z}_7$ and $\overline{Z}_9$, are depicted in Figs. 6.3-6.7 and the divergent contributions are shown in Tables 6.2-6.4 suppressing factors of $(y\overline{y}L)^2$. The diagrams contributing to $\overline{Z}_4, \overline{Z}_7$ and $\overline{Z}_9$ cancel in pairs as they must due to the fact that, for instance, the diagrams contributing to $\overline{Z}_4$ have a one-loop $\phi^2\overline{\phi}^2$ subdiagram and, in the uneliminated case, there is no counterterm for such a divergence.

	$\overline{Z}_2$	$\overline{Z}_5$
a	$12k_3$	$6k_3$
b	$6Ik_3$	$3Ik_3$
c	$6k_3$	$2(k_3+k_6)$
d	$2Ik_2$	Ik_2
e	Ik_2	$\frac{1}{2}Ik_2$
f	$2Ik_2$	Ik_2
g	$4Ik_2$	$2Ik_2$
h	$8k_2$	$4k_2$
i	k_2	k_5
j	$2Ik_2$	Ik_2
k	$3Ik_3$	$I(k_3+k_6)$
l	$2Ik_2$	Ik_2
m	$\frac{15}{2}k_4$	$\frac{3}{2}(2k_4+k_7)$
n	$4k_1$	$2k_1$
o	$2Ik_1$	Ik_1
p	$4Ik_1$	$2Ik_1$
q	$2Ik_1$	Ik_1
r	$4Ik_2$	$2Ik_2$
s	$4k_2$	$2k_2$
t	$2k_2$	k_2
u	$-Ik_2$	$-\frac{1}{2}Ik_2$

Table 6.3: Divergent contributions from Figs. 6.4, 6.5

	$\overline{Z}_3$	$\overline{Z}_6$	$\overline{Z}_8$
a	$8Ik_2$	$8Ik_2$	$2Ik_2$
b	$6Ik_3$	$I(5k_3+2k_6)$	$I(k_3+k_6)$
c	$2Ik_3$	$I(k_3+2k_6)$	Ik_6
d	$2Ik_3$	$I(k_3+2k_6)$	Ik_6
e	$2Ik_3$	$I(k_3+2k_6)$	Ik_6
f	$4Ik_1$	$4Ik_1$	Ik_1
g	$2Ik_2$	$I(k_2+2k_5)$	Ik_5
h	$4Ik_k$	$2I(k_2+2k_5)$	$2Ik_5$
i	$2Ik_2$	$I(k_2+2k_5)$	Ik_5
j	$4Ik_1$	$4Ik_1$	Ik_1
k	$\frac{5}{2}$	$I(k_4+3k_7)$	$I(k_9+k_7)$
l	$10Ik_4$	$I(7k_4+6k_7)$	$I(k_4+3k_7)$
m	$-6Ik_3$	$-I(5k_3+2k_6)$	$-I(k_3+k_6)$
n	$-4Ik_1$	$-4Ik_1$	$-Ik_1$
o	$12Ik_3$	$2I(5k_3+2k_6)$	$2I(k_3+k_6)$
p	$4Ik_2$	$4Ik_2$	Ik_2
q	$-8Ik_2$	$-8Ik_2$	$-2Ik_2$
r	$8Ik_2$	$8Ik_2$	$2Ik_2$
s	$4Ik_3$	$2I(k_3+2k_6)$	$2Ik_6$
t	$6Ik_3$	$I(5k_3+2k_6)$	$I(k_3+k_6)$

Table 6.4: Divergent contributions from Fig. 6.6, 6.7

The total two-loop divergences are given by

$$\begin{aligned}
\overline{Z}_1^{(2)} &= (y\overline{y}L)^2 \left\{ k_1 + \left[\frac{1}{2} I + 2\right] k_2 + \frac{3}{2} k_3 \right\}, \\
\overline{Z}_2^{(2)} &= (y\overline{y}L)^2 \left\{ 4\,[2I+1]\, k_1 + [16I+15]\, k_2 + 9\,[I+2]\, k_3 + \frac{15}{2} k_4 \right\}, \\
\overline{Z}_3^{(2)} &= (y\overline{y}L)^2 \left\{ \left[4k_1 + 20k_2 + 28k_3 + \frac{25}{2} k_4 \right] I \right\}, \\
\overline{Z}_4^{(2)} &= 0, \\
\overline{Z}_5^{(2)} &= (y\overline{y}L)^2 \left\{ 2\left([2I+1]\, k_1 + \left[4I + \frac{7}{2}\right] k_2 + 2\,[I+2]\, k_3 + \frac{3}{2} k_4 \right) \right. \\
&\qquad \left. + k_5 + [I+2]\, k_6 + \frac{3}{2} k_7 \right\}, \\
\overline{Z}_6^{(2)} &= (y\overline{y}L)^2 \left\{ 2\,(2k_1 + 8k_2 + 10k_3 + 4k_4) + 8k_5 + 16k_6 + 9k_7 \right\} I, \\
\overline{Z}_7^{(2)} &= 0, \\
\overline{Z}_8^{(2)} &= (y\overline{y}L)^2 \left\{ k_1 + 3k_2 + 3k_3 + k_4 + 4\,(k_5 + 2k_6 + k_7) + k_9 \right\} I, \\
\overline{Z}_9^{(2)} &= 0.
\end{aligned} \tag{6.80}$$

From the bare form of Eq. (6.28) and using Eqs. (6.41), (6.42), the two-loop contributions to the bare couplings are given by

$$\begin{aligned}
k_{1B}^{(2)} &= 6\overline{Z}_1^{(2)} - 3Z^{(2)} k_1 - 3\left(Z^{(1)}\right)^2 k_1 - 3Z^{(1)}\left(k_{1B}^{(1)} + 3R_B^{(1)}\right) - 3R_B^{(2)}, \\
k_{2B}^{(2)} &= 2\overline{Z}_2^{(2)} - 2Z^{(2)} k_2 - \left(Z^{(1)}\right)^2 k_2 - 2Z^{(1)}\left(k_{2B}^{(1)} - R_B^{(1)} + 2S_B^{(1)}\right) \\
&\quad + R_B^{(2)} - 2S_B^{(2)}, \\
k_{3B}^{(2)} &= 2\overline{Z}_3^{(2)} - Z^{(2)} k_3 - Z^{(1)}\left(k_{3B}^{(1)} - 2S_B^{(1)} + T_B^{(1)}\right) + 2S_B^{(2)} - T_B^{(2)}, \\
k_{4B}^{(2)} &= 6\overline{Z}_4^{(2)} + 3T_B^{(2)}, \\
k_{5B}^{(2)} &= 2\overline{Z}_5^{(2)} - 2Z^{(2)} k_5 - \left(Z^{(1)}\right)^2 k_5 - 2Z^{(1)}\left(k_{5B}^{(1)} + 2U_B^{(1)}\right) - 2U_B^{(2)}, \\
k_{6B}^{(2)} &= \overline{Z}_6^{(2)} - Z^{(2)} k_6 - Z^{(1)}\left(k_{6B}^{(1)} - U_{1B}^{(1)} + V_B^{(1)}\right) + U_B^{(2)} - V_B^{(2)}, \\
k_{7B}^{(2)} &= 2\overline{Z}_7^{(2)} + 2V_B^{(2)}, \\
k_{8B}^{(2)} &= \overline{Z}_8^{(2)} - Z^{(2)} k_8 - Z^{(1)}\left(k_{8B}^{(1)} + W_B^{(1)}\right) - W_B^{(2)}, \\
k_{9B}^{(2)} &= \overline{Z}_9^{(2)} + W_B^{(2)}
\end{aligned} \tag{6.81}$$

which, using Eqs. (6.43), (6.50), (6.77), (6.80), gives

$$
\begin{aligned}
k_{1B}^{(2)} &= \Big[3\,(I+4)\,k_1 + 3\,(I+7)\,k_2 + 9k_3\Big]\,(y\overline{y}L)^{\,2} - 3R_B^{(2)}\,, \\
k_{2B}^{(2)} &= \Big[16\,(I+1)\,k_1 + (34I+49)\,k_2 + 6\,(3I+8)\,k_3 + 15k_4\Big]\,(y\overline{y}L)^{\,2} \\
&\quad + R_B^{(2)} - 2S_B^{(2)}\,, \\
k_{3B}^{(2)} &= \Big[8Ik_1 + 4\,(10I+1)\,k_2 + 3\,(19I+3)\,k_3 + 5\,(5I+1)\,k_4\Big]\,(y\overline{y}L)^{\,2} \\
&\quad + 2S_B^{(2)} - T_B^{(2)}\,, \\
k_{4B}^{(2)} &= 3T_B^{(2)}\,, \\
k_{5B}^{(2)} &= \Big[8\,(I+1)\,k_1 + 2\,(8I+11)\,k_2 + 4\,(2I+5)\,k_3 + 6k_4 + (2I+5)\,k_5 \\
&\quad + 2\,(I+4)\,k_6 + 3k_7\Big]\,(y\overline{y}L)^{\,2} - 2U_B^{(2)}\,, \\
k_{6B}^{(2)} &= \Big[4Ik_1 + (16I+1)\,k_2 + 2\,(10I+1)\,k_3 + (8I+1)\,k_4 + 2\,(4I+1)\,k_5 \\
&\quad + (17I+5)\,k_6 + 3\,(3I+1)\,k_7\Big]\,(y\overline{y}L)^{\,2} + U_B^{(2)} - V_B^{(2)}\,, \\
k_{7B}^{(2)} &= 2V_B^{(2)}\,, \\
k_{8B}^{(2)} &= \Big[I\,(k_1 + 3k_2 + 3k_3 + k_4) + (4I+1)\,(k_5 + 2k_6 + k_7) \\
&\quad + (I+1)\,(k_8 + k_9)\Big]\,(y\overline{y}L)^{\,2} - W_B^{(2)}\,, \\
k_{9B}^{(2)} &= W_B^{(2)}\,.
\end{aligned} \tag{6.82}
$$

As at one loop, requiring that k_{iB} as given by Eqs. (6.43), (6.44) be μ-independent gives

$$\beta_i^{(2)} = 2\kappa^{(2,1)} \tag{6.83}$$

together with the consistency conditions for the two-loop double poles:

$$
\begin{aligned}
2\left(16\pi^2\right)\kappa_1^{(2,2)} &= 3\left(\kappa_1^{(1,1)} + \kappa_2^{(1,1)}\right)y\overline{y} + 3\left(\beta_y^{(1)}\overline{y} + y\beta_{\overline{y}}^{(1)}\right)(k_1+k_2)\,, \\
2\left(16\pi^2\right)\kappa_2^{(2,2)} &= 2\left(2\kappa_1^{(1,1)} + 5\kappa_2^{(1,1)} + 3\kappa_3^{(1,1)}\right)y\overline{y} \\
&\quad + 2\left(\beta_y^{(1)}\overline{y} + y\beta_{\overline{y}}^{(1)}\right)(2k_1 + 5k_2 + 3k_3) \quad \text{etc.}
\end{aligned} \tag{6.84}
$$

From Eqs. (6.43), (6.44), (6.81), (6.83), the two-loop β-functions are given by

$$
\begin{aligned}
\beta_1^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} 3\left(k_1 + k_2 + 2r_2\right), \\
\beta_2^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} 2\left(8k_1 + 17k_2 + 9k_3 - r_2 + 2s_2\right), \\
\beta_3^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} \left(8k_1 + 40k_2 + 57k_3 + 25k_4 - 4s_2 + 2t_2\right), \\
\beta_4^{(2)} &= \frac{(y\overline{y})^2}{(16\pi^2)^2} 6t_2, \\
\beta_5^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} 2\left(4\left[k_1 + 2k_2 + k_3\right] + k_5 + k_6 + 2u_2\right), \\
\beta_6^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} \left(4k_1 + 16k_2 + 20k_3 + 8K_4 + 8k_5 + 17k_6 + 9k_7 - 2u_2 + 2v_2\right), \\
\beta_7^{(2)} &= 4\frac{(y\overline{y})^2}{(16\pi^2)^2} v_2, \\
\beta_8^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} \left(k_1 + 3k_2 + 3k_3 + k_4 + 4\left[k_5 + 2k_6 + k_7\right] + k_8 + k_9 + 2w_2\right), \\
\beta_9^{(2)} &= \frac{(y\overline{y})^2}{(16\pi^2)^2} 2w_2,
\end{aligned}
\tag{6.85}
$$

where $r_2 = \frac{r^{(2,1)}}{(16\pi^2)^2}$ etc.

We can check, using Eqs. (6.43), (6.44), (6.49), (6.50), (6.81), that Eq. (6.84) is satisfied provided we take

$$
\begin{aligned}
r^{(2,2)} &= \frac{(y\overline{y})^2}{(16\pi^2)^2} \left(r_1 + s_1\right), \\
s^{(2,2)} &= \frac{(y\overline{y})^2}{(16\pi^2)^2} \frac{1}{2} \left(r_1 + 5s_1 + 3t_1\right), \\
t^{(2,2)} &= 0, \\
u^{(2,2)} &= \frac{(y\overline{y})^2}{(16\pi^2)^2} \frac{1}{2} \left(r_1 + 2s_1 + t_1 + u_1 + v_1\right), \\
v^{(2,2)} &= 0, \\
w^{(2,2)} &= 0.
\end{aligned}
\tag{6.86}
$$

Due to the arbitrariness of the non-linear renormalisations of $\overline{F}$, there is no obvious way of verifying these double pole relations by direct calculation.

6.3.5 Eliminated Two-Loop Renormalisation

Examples of the classes of two-loop diagrams in the eliminated case are depicted in Fig. 6.8, and the corresponding divergent contributions are listed in Table 6.5. We find

$$
\begin{aligned}
Y_1^{(2)} &= (y\overline{y}L)^2 \left[\frac{25}{2} I + 10\right] \lambda_1, \\
Y_2^{(2)} &= (y\overline{y}L)^2 \left[2I + 1\right] \left[4\lambda_1 + \frac{9}{2}\lambda_2\right], \\
Y_3^{(2)} &= (y\overline{y}L)^2 \left\{I\lambda_1 + \left[4I + 1\right]\lambda_2 + \left[I + 1\right]\lambda_3\right\}.
\end{aligned}
\tag{6.87}
$$

	$\overline{G}^3$	$\overline{m}^2\overline{G}^2$	$\overline{m}^4\overline{G}$
a	$-5I\lambda_1$	$-\frac{9}{2}I\lambda_1 - 6I\lambda_2$	$-I(\lambda_1 + 5\lambda_2 + 2\lambda_3)$
b	$5I\lambda_1$	$\frac{9}{2}I\lambda_1 + 6I\lambda_2$	$I(\lambda_1 + 5\lambda_2 + 2\lambda_3)$
c	$10I\lambda_1$	$7I\lambda_1 + 6I\lambda_2$	$I(\lambda_1 + 3\lambda_2)$
d	$\frac{5}{2}I\lambda_1$	$I\lambda_1 + 3I\lambda_2$	$I(\lambda_2 + \lambda_3)$
e	$\frac{5}{2}\lambda_1$	$\lambda_1 + 3\lambda_2$	$(\lambda_2 + \lambda_3)$
f	$\frac{15}{2}\lambda_1$	$3\lambda_1 + \frac{3}{2}\lambda_2$	0

Table 6.5: Divergent contributions from Fig. 6.8

Requiring μ-independence of λ_{iB} in Eq. (6.58) leads to

$$\beta_{\lambda_i}^{(2)} = 2L_i^{(2,1)} \tag{6.88}$$

and the consistency conditions for two-loop double poles:

$$\begin{aligned}
2\left(16\pi^2\right) L_1^{(2,2)} &= 15\left[L_1^{(1,1)} y\overline{y}\left(\beta_y^{(1)}\overline{y} + y\beta_{\overline{y}}^{(1)}\right)\lambda_1\right], \\
2\left(16\pi^2\right) L_2^{(2,2)} &= \left(2L_1^{(1,1)} + 6L_2^{(1,1)}\right) y\overline{y} + \left(\beta_y^{(1)}\overline{y} + y\beta_{\overline{y}}^{(1)}\right)(2\lambda_1 + 6\lambda_2), \\
2\left(16\pi^2\right) L_3^{(2,2)} &= \left(L_2^{(1,1)} + L_3^{(1,1)}\right) y\overline{y} + \left(\beta_y^{(1)}\overline{y} + y\beta_{\overline{y}}^{(1)}\right)(\lambda_2 + \lambda_3).
\end{aligned} \tag{6.89}$$

This leads, via Eqs. (6.56), (6.58), (6.59), (6.87), to β-functions

$$\begin{aligned}
\beta_{\lambda_1}^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} 75\lambda_1, \\
\beta_{\lambda_2}^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2} 2(8\lambda_1 + 9\lambda_2), \\
\beta_{\lambda_3}^{(2)} &= -\frac{(y\overline{y})^2}{(16\pi^2)^2}(\lambda_1 + 4\lambda_2 + \lambda_3),
\end{aligned} \tag{6.90}$$

and we can check, using Eqs. (6.49), (6.56), (6.58), (6.59), (6.60), (6.87), that Eq. (6.89) is satisfied.

As at one loop, Eq. (6.62) is crucial for consistency between the uneliminated and eliminated formalisms, and leads, via Eqs. (6.45), (6.56), (6.81), to

$$\begin{aligned}
\frac{1}{6}\lambda_{1B}^{(2)} &= \overline{Z}_1^{(2)} + \overline{Z}_2^{(2)} + \overline{Z}_3^{(2)} + \overline{Z}_4^{(2)} - \frac{1}{2}Z^{(2)}(k_1 + 2k_2 + k_3) \\
&\quad - Z^{(1)}\left(3\overline{Z}_1^{(1)} + 2\overline{Z}_2^{(1)} + \overline{Z}_3^{(1)}\right) + \frac{1}{2}\left(Z^{(1)}\right)^2(2k_1 + 3k_2 + k_3), \\
\frac{1}{2}\lambda_{2B}^{(2)} &= \overline{Z}_5^{(2)} + \overline{Z}_6^{(2)} + \overline{Z}_7^{(2)} - Z^{(2)}(k_5 + k_6) - Z^{(1)}\left(2\overline{Z}_5^{(1)} + \overline{Z}_6^{(1)}\right) \\
&\quad + \frac{1}{2}\left(Z^{(1)}\right)^2(3k_5 + 2k_6), \\
\lambda_{3B}^{(2)} &= \overline{Z}_8^{(2)} + \overline{Z}_9^{(2)} - Z^{(2)}k_8 - Z^{(1)}\overline{Z}_8^{(1)} + \left(Z^{(1)}\right)^2 k_8.
\end{aligned} \tag{6.91}$$

We can check with the aid of Eqs. (6.48), (6.38), (6.77), (6.80), (6.87) that this is satisfied.

As at one loop, if we solve Eq. (6.64), we find consistency conditions for λ_{1-3} which in this case are given by

$$\begin{aligned}(75+3\rho)\,\lambda_1 &= 0,\\ 16\lambda_1 + (18+3\sigma)\,\lambda_2 &= 0,\\ \lambda_1 + 4\lambda_2 + (1+3\tau)\,\lambda_3 &= 0.\end{aligned} \tag{6.92}$$

Again, these conditions are equivalent to the equations we would have derived using Eqs. (6.90) had we sought a simlar renormalisation group-invariant solution in the eliminated form of the theory. As at one loop we see, using Eq. (6.79), that imposing either the third or fourth special cases mentioned previously (Eqs. (6.73), (6.74)) will again satisfy Eq. (6.64) with $r_1 = s_1 = t_1 = u_1 = v_1 = w_1 = 0$.

6.4 Comparison of Formalisms

As this work was inspired by similar work in Ref. [89], in which the calculations were performed in the superfield formalism, we compare our results with theirs in order to check for consistency. In fact, we find differences in the two sets of results which require explanation. Allowing for differences in notation between the two calculations, rewriting our parameters as

$$\begin{array}{lll} k_1 \to 2g\overline{g}\,(\det C)\,, & k_5 \to 8\overline{g}^2 k_2'\,(\det C)\,, & k_8 \to 8\overline{g}^3 k_1'\,(\det C)\,,\\ \epsilon \to 2\epsilon, & y \to 2g, & \overline{y} \to 2\overline{g} \end{array} \tag{6.93}$$

(where the primed couplings are those appearing in the superfield calculation), and setting our remaining k_i's to zero as they do not appear in Ref. [89], we find that the two sets of results agree precisely at one loop. However, at two loops, differences appear in the β-functions. We would expect that the results we derived in the eliminated formalism would be equivalent to the superfield calculation when rewriting

$$\lambda_1 \to g\overline{g}\gamma. \tag{6.94}$$

However, it is the β-functions for λ_1 and γ that are found to disagree at two loops. Whilst β-functions are generally renormalisation scheme-dependent beyond one loop, this does not explain the present discrepancy as the agreement between the two calculations for individual diagrams demonstrates that both calculations use the same scheme. The difference is in fact due to the way in which we have identified F with $\overline{G}$. In our case, we made this identification by eliminating the auxiliary fields by means of their equations of motion (Eq. (6.52)). We have demonstrated in Eq. (6.62) that elimination of the auxiliary fields from the renormalised theory is equivalent to renormalisation of the eliminated classical action so this identification is valid beyond the classical case and our procedure is consistent. The identification of F with $\overline{G}$ in Ref. [89], however, follows a different procedure in which the identification is corrected at each order. That is, F and $\overline{G}$ are identified immediately such that there is only a single coupling for the F^3 term and no independent copulings for terms of the form $F^n\overline{G}^{3-n}$ are introduced with additional correction to this identification at higher orders. In addition, the precise form of the identification is of $\overline{G}^2$ with $\frac{1}{2}F^2$ and $\overline{G}^3$ with $\frac{1}{6}F^3$. In effect, the authors of [89] have made the replacements

$$\begin{aligned}&\overline{Z}_1 \to \overline{Z}_1 + \overline{Z}_2 + \frac{1}{2}\overline{Z}_3 + \frac{1}{6}\overline{Z}_4,\\ &\overline{Z}_i \to 0, \qquad i = 2,3,4,\\ &k_i \to 0, \qquad i = 2,3,4.\end{aligned} \tag{6.95}$$

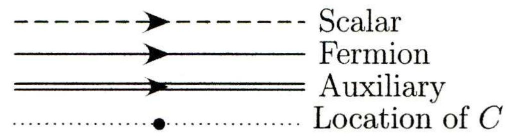


Figure 6.1: Conventions for Feynman diagrams in the Wess-Zumino model

In the eliminated case, the two sets of results agree on the form of the two-loop double poles in $\lambda_{1B}^{(2)}$. In the superfield calculation, having started from the uneliminated calculation, $\lambda_{1B}^{(2)}$ is derived by assuming the relations between the eliminated and uneliminated parameters (Eq. (6.62)). After applying Eq. (6.95), the result for the double poles is the same as in our calculation (Eq. (6.91)). The $2k_1$ term of $\overline{Z}_2^{(2)}$ in Eq. (6.38) is to $\overline{Z}_1^{(2)}$ and $Z_3^{(2)}$ is reduced by a factor of 2 through Eq. (6.95). These two alterations have no effect in the case of the double poles leading to agreement between the two calculations. However, as they do have an effect on the single poles, there is no agreement in this case. The different means of identifying F and $\overline{G}$ appears to cause problems with the identification of λ_1 and γ at higher orders. There is, therefore, no obvious means of translating between the two sets of results beyond one loop.

6.5 Summary

We have shown that it is necessary to include in the action from the outset a full set of terms that may be generated through renormalisation, each with its own coupling. In this way, we may achieve agreement between calculations performed in the eliminated and uneliminated formalisms. That is, we find that the results are equivalent if the auxiliary fields are eliminated from the renormalised action or if the eliminated classical action is renormalised. In particular, the forms of the action with renormalisation group-invariant trajectories for the couplings agree in both formalisms. We have also demonstrated the existence of two particular forms for the action which require no non-linear renormalisation of the auxiliary fields at least to two loops. As we shall see in the next chapter, these solutions are also of interest in the $U(1)$-gauged model. In addition, we have now provided an explicit example of a model which is power-counting non-renormalisable but which, thanks to the symmetries of Eqs. (6.33), (6.34), has a UV-finite effective action in the manner alluded to in §4.3.

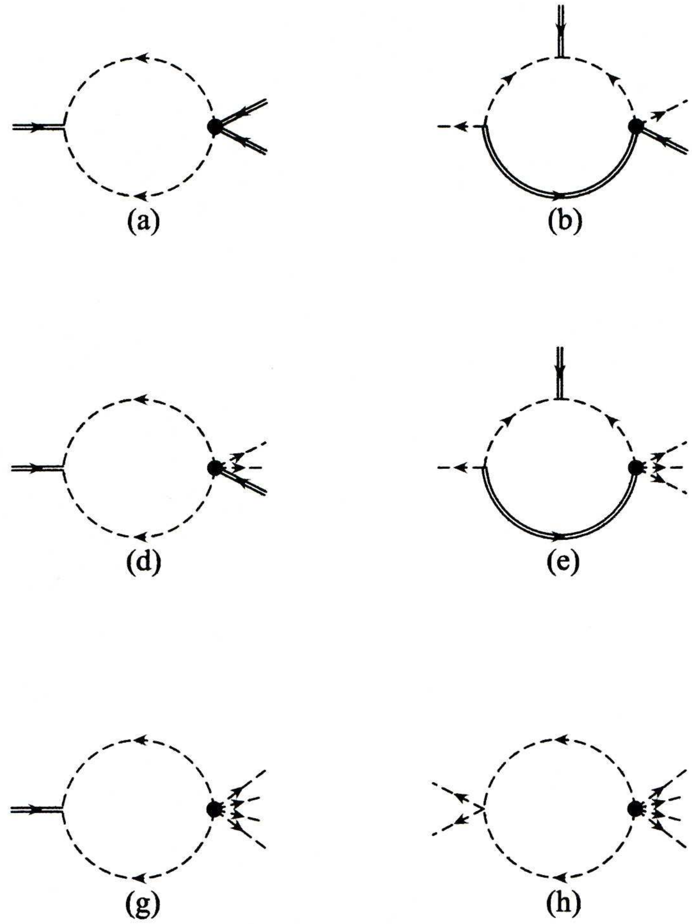


Figure 6.2: One-loop diagrams

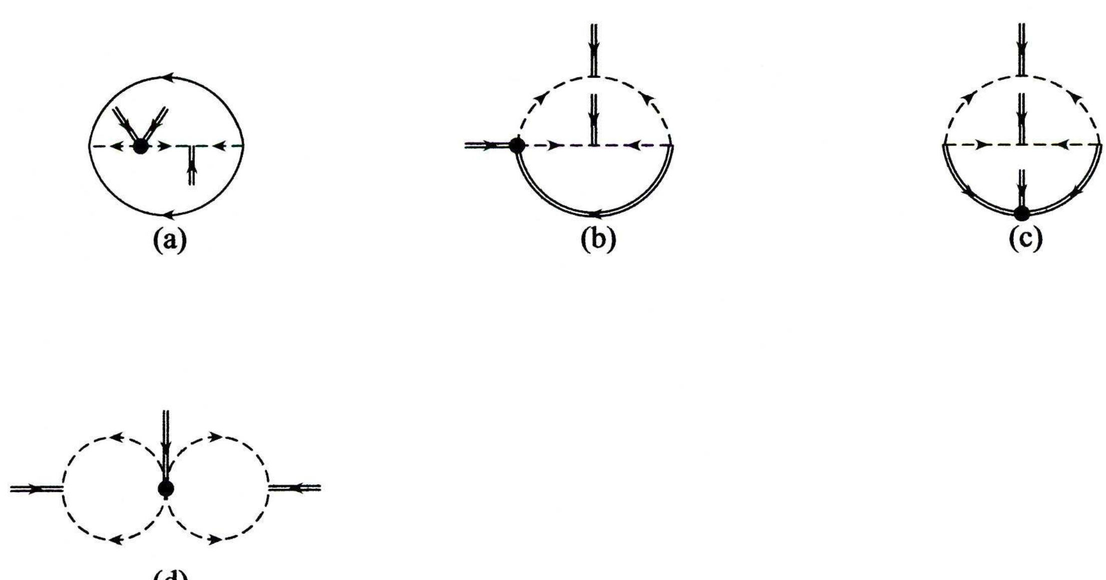


Figure 6.3: Two-loop F^3 diagrams

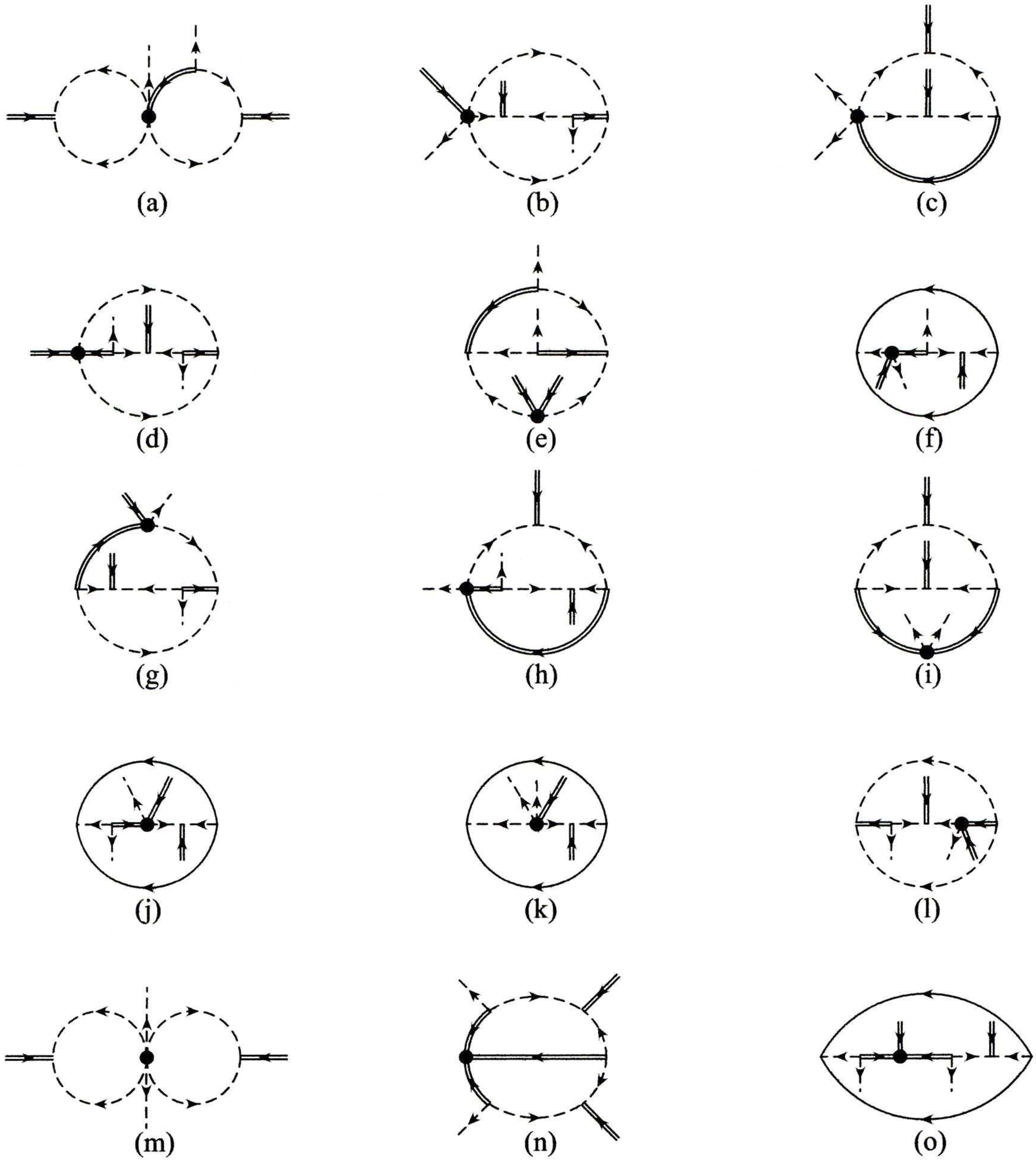


Figure 6.4: Two-loop $F^2\overline{\phi}^2$ diagrams

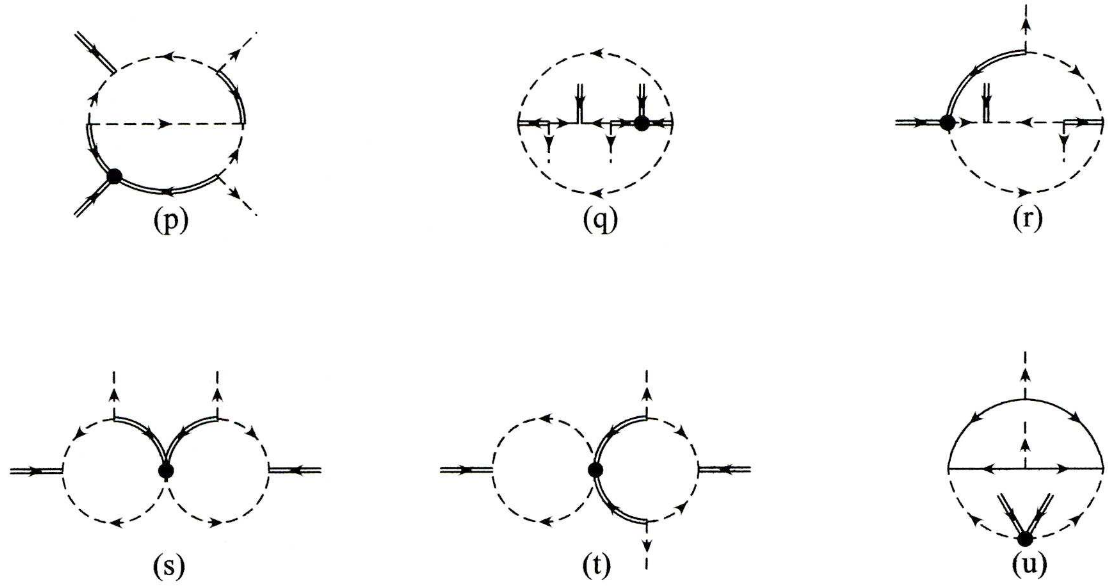


Figure 6.5: Two-loop $F^2\overline{\phi}^2$ diagrams (continued)

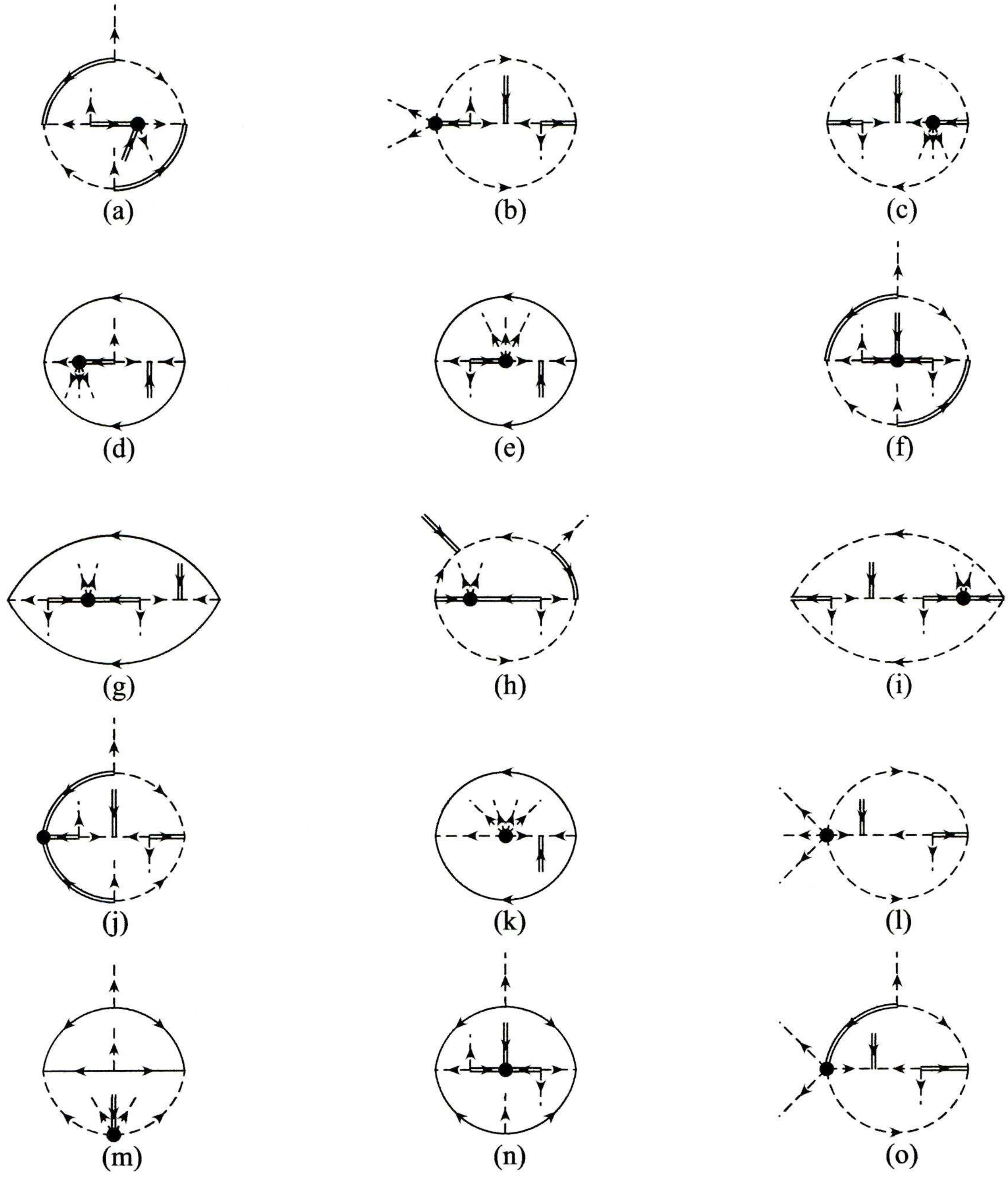


Figure 6.6: Two-loop $F\overline{\phi}^4$ diagrams

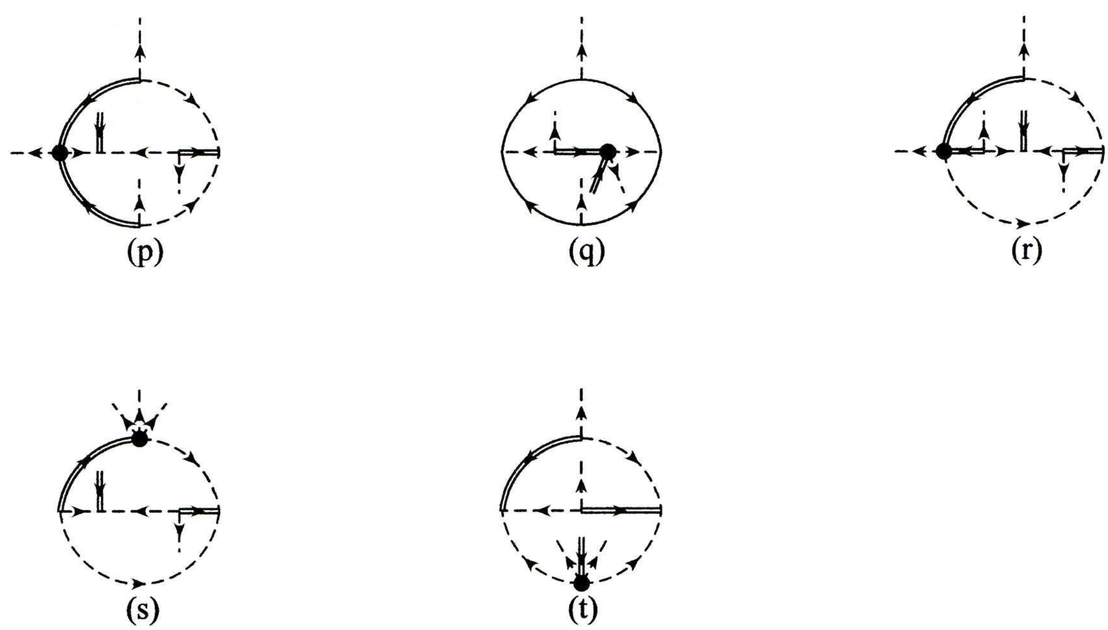


Figure 6.7: Two-loop $F\overline{\phi}^4$ diagrams (continued)

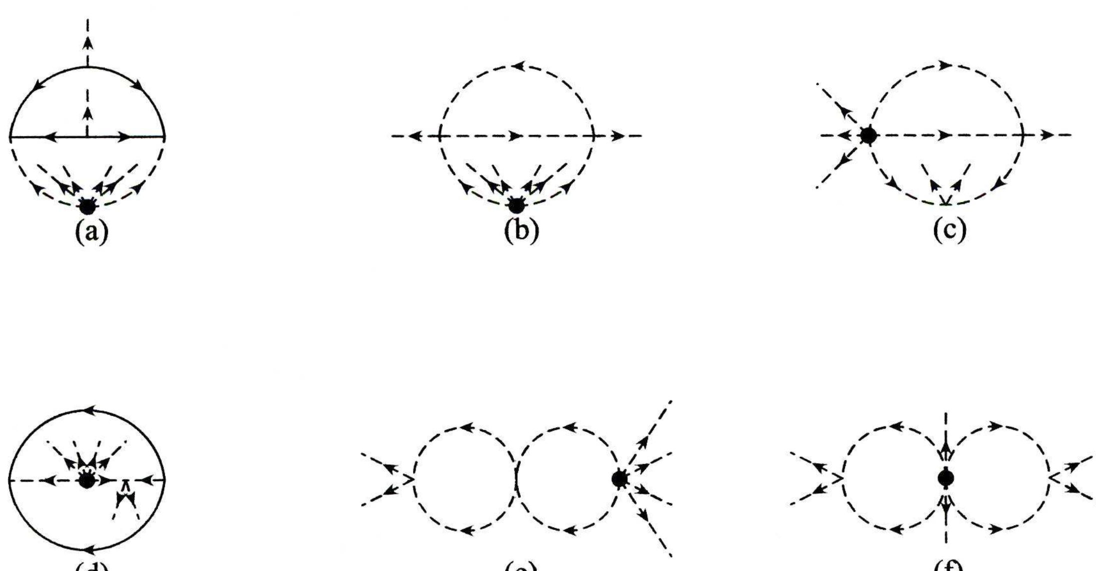


Figure 6.8: Two-loop $\overline{\phi}^6$ diagrams in the eliminated case

Chapter 7

Deformed $U(1)$ Gauge Theory

Having focused on the simple Wess-Zumino model in the previous chapter, we now wish to introduce a gauge group to the model. Non-anticommutative pure gauge theories have been studied in Ref. [50] and theories with non-interacting matter in Ref. [51, 60] so here we wish to retain the superpotential whilst introducing a gauge symmetry. It has been shown in Ref. [54] that there are constraints on possible gauge groups in the non-anticommutative case. These constraints appear to rule out the possibility of introducing a non-abelian gauge group in a theory with a superpotential, although we will return to this topic in the next chapter when we discuss the constraints in more detail as well as a means to circumvent them. In the present case, we consider an abelian ($U(1)$) gauge group [93]. The simplest possible model with a trilinear superpotential requires three chiral fields in order for the $U(1)$ charges to obey relations following from gauge-invariance and anomaly cancellation as will become clear. We take the chiral fields to be massless for simplicity as the effect of masses on non-anticommutative theories has already been given a comprehensive treatment in Ref. [52].

7.1 The Non-anticommutative $U(1)$ Action

In this section we will give the form of the action for an $\mathcal{N} = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory with a superpotential as discussed in Refs. [37, 47, 69, 94]. We take the components of the chiral multiplet to be ϕ_i, ψ_i, F_i, $i = 1, 2, 3$, with corresponding $U(1)$ charges q_i, $i = 1, 2, 3$. We split the action into kinetic and potential terms:

$$S_0 = S_{\text{kin}} + S_{\text{pot}} \tag{7.1}$$

with

$$\begin{aligned} S_{\text{kin}} = \int d^4x \Big[& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\overline{\lambda}\overline{\sigma}^{\mu}\left(D_{\mu}\lambda\right) + \frac{1}{2}D^2 \\ & - igC^{\mu\nu}F_{\mu\nu}\overline{\lambda}\overline{\lambda} + \overline{F}_i F_i - i\overline{\psi}_i\overline{\sigma}^{\mu}\left(D_{\mu}\psi\right)_i - \left(D^{\mu}\overline{\phi}\right)_i\left(D_{\mu}\phi\right)_i \\ & + \sqrt{2}gC^{\mu\nu}\left(D_{\mu}\overline{\phi}\right)_i\overline{\lambda}\overline{\sigma}_{\nu}\psi_i + igC^{\mu\nu}\overline{\phi}_i F_{\mu\nu}F_i + \frac{1}{4}|C|^2 g^2 \overline{F}_i\overline{\phi}_i\overline{\lambda}\overline{\lambda} \\ & + \sum_i \Big\{ gq_i\overline{\phi}_i D\phi_i + i\sqrt{2}gq_i\left(\overline{\phi}_i\lambda\psi_i - \overline{\psi}_i\overline{\lambda}\phi_i\right) \\ & - \gamma_i C^{\mu\nu}g\left[\sqrt{2}\left(D_{\mu}\overline{\phi}\right)_i\overline{\lambda}\overline{\sigma}_{\nu}\psi_i + \sqrt{2}\overline{\phi}_i\overline{\lambda}\overline{\sigma}_{\nu}\left(D_{\mu}\psi\right)_i + i\overline{\phi}_i F_{\mu\nu}F_i\right]\Big\}\Big], \end{aligned} \tag{7.2}$$

where the gauge-covariant derivative is defined by

$$(D_\mu\phi) = (\partial_\mu + igq_iA_\mu)\,\phi_i. \tag{7.3}$$

It should be noted that the terms multiplied by γ_i are not found in the classical action derived from the reduction to components of the superspace action but are included here as they are found to be required for renormalisability.

For the potential term, we take a superpotential of the form $W(\Phi) = y\Phi_1\Phi_2\Phi_3$ leading to

$$\begin{aligned} S_{\text{pot}} = -\int \mathrm{d}^4x \Big[&\{(F_iG_i - y\phi_1\psi_2\psi_3 - y\phi_2\psi_3\psi_1 - y\phi_3\psi_1\psi_2) + \text{h.c.}\} \\ &+ 2ig\overline{y}C^{\mu\nu}F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3 - \frac{1}{4}y|C|^2F_1F_2F_3\Big], \end{aligned} \tag{7.4}$$

where

$$G_1 = y\phi_2\phi_3 \tag{7.5}$$

with similar expressions for G_2 and G_3.

For gauge invariance of S_{pot} we require

$$q_1 + q_2 + q_3 = 0, \tag{7.6}$$

and for anomaly cancellation we require

$$q_1q_2q_3 = 0. \tag{7.7}$$

Assuming that we have at least one charged field, these constraints are the reason for requiring a minimum of three chiral fields. Furthermore, they restrict us to the set of charges $(q, -q, 0)$; that is, with only three fields, we must consider a positive, a negative and a neutral field under the $U(1)$ symmetry. A cubic term in the neutral field would not break gauge invariance and so, in fact, the most general superpotential we could consider is of the form

$$W(\Phi) = y\Phi_1\Phi_2\Phi_3 + y'\Phi_3^3, \tag{7.8}$$

where we assume Φ_3 to be the neutral field. However, we choose to neglect cubic terms of this form and focus purely on the mixed term allowing us to express formulae in a form which is symmetric under permutations of q_i.

Following the procedure of the preceding chapters, we can show that the SUSY transformations of the component fields are given by

$$\begin{aligned} \delta_\epsilon A_\mu &= -i\overline{\lambda}\overline{\sigma}_\mu\epsilon, \\ \delta_\epsilon\lambda_\alpha &= i\epsilon_\alpha D + (\sigma^{\mu\nu}\epsilon)_\alpha\left[F_{\mu\nu} + \frac{1}{2}iC_{\mu\nu}\overline{\lambda\lambda}\right], \\ \delta_\epsilon\overline{\lambda}_{\dot\alpha} &= 0, \\ \delta_\epsilon D &= -\epsilon\sigma^\mu D_\mu\overline{\lambda}, \\ \delta_\epsilon\phi_i &= \sqrt{2}\epsilon\phi_i, \\ \delta_\epsilon\overline{\phi}_i &= 0, \\ \delta_\epsilon\psi_i^\alpha &= \sqrt{2}\epsilon^\alpha F_i, \\ \delta_\epsilon\overline{\psi}_{i\dot\alpha} &= -i\sqrt{2}\left(D_\mu\overline{\phi}_i\right)(\epsilon\sigma^\mu)_{\dot\alpha}, \\ \delta_\epsilon F_i &= 0, \\ \delta_\epsilon\overline{F}_i &= -i\sqrt{2}D_\mu\overline{\psi}_i\overline{\sigma}^\mu\epsilon - 2igq_i\overline{\phi}_i\epsilon\lambda + 2C^{\mu\nu}gD_\mu\left(\overline{\phi}_i\epsilon\sigma_\nu\overline{\lambda}\right). \end{aligned} \tag{7.9}$$

In order to find all terms that can be generated by renormalisation, we must consider the symmetries of the classical action. We have, as in the ungauged case, a pseudo-R symmetry under

$$\begin{aligned}\phi_i &\to e^{-i\alpha}\phi_i,\\ F_i &\to e^{i\alpha}F_i,\\ \lambda &\to e^{-i\alpha}\lambda,\\ C^{\alpha\beta} &\to e^{-2i\alpha}C^{\alpha\beta},\\ y &\to e^{i\alpha}y,\end{aligned} \tag{7.10}$$

and a pseudo-chiral symmetry under

$$\begin{aligned}\phi_i &\to e^{i\gamma}\phi_i,\\ y &\to e^{-3i\gamma}y,\end{aligned} \tag{7.11}$$

where barred fields transform with opposite charges and the gauge fields are unaffected.

The new divergent terms that can arise subject to these symmetries consist of

$$|C|^2F^2\overline{\phi}^2, \qquad \overline{y}|C|^2F\overline{\phi}^4, \qquad \overline{y}^2|C|^2\overline{\phi}^6, \qquad \overline{y}|C|^2\overline{\lambda}\overline{\lambda}\phi^3. \tag{7.12}$$

Note that terms of the form $\overline{\phi}^2\psi(C\psi)$ are also allowed by the above symmetries but are not $\mathcal{N}=\frac{1}{2}$-invariant. In addition, the combination

$$\overline{y}^{-1}\left[F_1\psi_2\left(C\psi_3\right)+F_2\psi_3\left(C\psi_1\right)+F_3\psi_1\left(C\psi_2\right)\right] \tag{7.13}$$

is also allowed by the above symmetries and is $\mathcal{N}=\frac{1}{2}$-invariant. However, as this term is not generated as a one-loop divergence unless already present in the classical action, we choose to omit it.

Including these new terms, each with their own coefficient, we are led to the complete action:

$$S = S_0 + S_{\text{gen}}, \tag{7.14}$$

where S_0 is given in (7.1) and

$$\begin{aligned}S_{\text{gen}} = \int \mathrm{d}^4x \Big[\overline{y}^{-1}|C|^2\Big\{ &\left(k_1-\frac{1}{4}y\overline{y}\right)F_1F_2F_3\\ &+k_2\left(F_1F_2\overline{G}_3+F_2F_3\overline{G}_1+F_3F_1\overline{G}_2\right)\\ &+k_3\left(F_1\overline{G}_2\overline{G}_3+F_2\overline{G}_3\overline{G}_1+F_3\overline{G}_1\overline{G}_2\right)+k_4\overline{G}_1\overline{G}_2\overline{G}_3\Big\}\\ &+|C|^2\left\{\left(K_1-\frac{1}{4}g^2\right)F_i\overline{\phi}_i+K_2\overline{y}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3\right\}\overline{\lambda}\overline{\lambda}\Big].\end{aligned} \tag{7.15}$$

As the $F_1F_2F_3$ and $F_i\overline{\phi}_i\overline{\lambda}\overline{\lambda}$ terms of the classical action and all the terms of S_{gen} are separately $\mathcal{N}=\frac{1}{2}$-invariant and $\mathcal{N}=\frac{1}{2}$-invariance will not preserve the values of their coefficients in the classical action, there is no reason for their renormalisation to be accounted for purely by replacing quantities in S_0 by bare quantities. As such, both terms as well as the new terms in S_{gen} are given their own arbitrary coefficients.

A final term that must be included in the action is the gauge-fixing term. The standard term

$$S_{\text{gf}} = \frac{1}{2\alpha}\int \mathrm{d}^4x\,(\partial\cdot A)^2. \tag{7.16}$$

a	$-2W_1$
b	W_1
c	$-W_1$
d	0

Table 7.1: Divergent contributions from Fig. 7.3

is not supersymmetric but as we are working in components we are nevertheless free to impose it without loss of formal supersymmetry [95]. However, as the supersymmetry of the model is no longer manifest due to this choice of gauge, we will find that the renormalisation constants for individual components of a supermultiplet differ.

The gauge propagator is then given by

$$\Delta_{\mu\nu} = -\frac{1}{p^2}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right) \tag{7.17}$$

and the fermion propagator by

$$\Delta_{\alpha\dot{\alpha}} = \frac{p_\mu \sigma^\mu_{\alpha\dot{\alpha}}}{p^2}, \tag{7.18}$$

where the momentum enters at the end of the propagator with the undotted index.

7.2 Renormalisation of the $U(1)$-Gauged Wess-Zumino Model

7.2.1 Renormalisation in the Uneliminated Case

The divergent contributions from one-loop diagrams to terms in S_{kin} can be extracted in part from results for the $SU(N)\otimes U(1)$ case as given in Refs. [50, 51]. These results are presented in total here without specifying individual contributions. However, in Ref. [54], the divergent contributions to S_{kin} arising from a superpotential are not given and so the $y\overline{y}$-dependent divergences are presented here. The diagrams which contribute to S_{kin} are depicted in Figs. 7.2, 7.3. As we now have gauge fields in addition to chiral multiplets, we must introduce additional notation for Feynman diagrams. The full set of our conventions for the diagrams in gauged theories is shown in Fig.7.1. The contribution from Fig. 7.2 is given by

$$-2\sqrt{2}y\overline{y}gLC^{\mu\nu}\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu\partial_\mu\psi_i \tag{7.19}$$

where, as before,

$$L = \frac{1}{16\pi^2\epsilon}. \tag{7.20}$$

The contributions from Fig. 7.3 are given in Table 7.1, where

$$W_1 = i\sqrt{2}y\overline{y}g^2LC^{\mu\nu}A_\mu\sum_i q_i\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu\psi_i. \tag{7.21}$$

Taking into account the contributions from Table 7.1, Eq. (7.19), and those which can be extracted from Ref. [51], we find the divergent contributions to the kinetic part of the action are given by

$$\begin{aligned}\Gamma^{\text{pole}}_{\text{kin}} = L\int d^4x\Big[& -2ig^3QC^{\mu\nu}F_{\mu\nu}\overline{\lambda}\overline{\lambda} - 2\sqrt{2}gy\overline{y}C^{\mu\nu}\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu D_\mu\psi_i \\ & + \sum_i\left(2\sqrt{2}\alpha g^3q_i^2C^{\mu\nu}D_\mu\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu\psi_i - 2ig^3C^{\mu\nu}q_i^2\overline{\phi}_iF_{\mu\nu}F_i\right)\Big].\end{aligned} \tag{7.22}$$

a	$4W_2 + 8W_3$
b	$4W_3$
c	$-2W_2 - 12W_3$
d	$8W_2$
e	$2\alpha W_2$
f	$2W_2$
g	$-4W_2 - 8W_3$
h	$8W_3$
i	$-2\alpha W_2$
j	$-2W_2$
k	$4W_2 + 8W_3$
l	$-8W_3$

Table 7.2: Divergent contributions from Fig. 7.4

The contributions to $S_{\rm pot}$ cannot be derived from earlier work in this way due to the different forms of the potential and so the diagrams contributing to the potential part of the action are shown in Fig. 7.4 and the resulting contributions are listed in Table 7.2, where

$$\begin{aligned} W_2 &= iQg^3 C^{\mu\nu} F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3, \\ W_3 &= ig^3 C^{\mu\nu}\left[q_1^2\partial_\mu\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3 + q_2^2\partial_\mu\overline{\phi}_2\overline{\phi}_3\overline{\phi}_1 + q_3^2\partial_\mu\overline{\phi}_3\overline{\phi}_1\overline{\phi}_2\right] A_\nu, \end{aligned} \tag{7.23}$$

and we employ the shorthand

$$Q = q_1^2 + q_2^2 + q_3^2. \tag{7.24}$$

The sum of the contributions from Table 7.2 is given by

$$10iQg^3 L \int d^4x \overline{y} C^{\mu\nu} F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3. \tag{7.25}$$

Note that the contributions of Figs. 7.4(e)-(h) cancel with those of Figs. 7.4(i)-(l). In fact, all pairs of diagrams that differ only by the insertion of an auxiliary field in this way will cancel and so similar sets of diagrams are omitted in the following.

The divergent contributions to the $F_1F_2F_3$ and $F_i\overline{\phi}_i\overline{\lambda}\overline{\lambda}$ terms are not included above as these terms now have individual couplings in $S_{\rm gen}$ and, as such, cannot be extracted from earlier work. For this reason, they will be covered in detail shortly. The remaining divergent contributions are denoted by

$$\begin{aligned} \Gamma^{\rm pole}_{\rm rem} = -\int d^4x \Big[|C|^2 \Big\{ & \overline{y}^{-1}\big[X_1F_1F_2F_3 + X_{2a}F_1F_2\overline{G}_3 + X_{2b}F_2F_3\overline{G}_1 \\ & + X_{2c}F_3F_1\overline{G}_2 + X_{3a}F_1\overline{G}_2\overline{G}_3 + X_{3b}F_2\overline{G}_3\overline{G}_1 \\ & + X_{3c}F_3\overline{G}_1\overline{G}_2 + X_4\overline{G}_1\overline{G}_2\overline{G}_3 + X_2'\left(F_1^2\overline{\phi}_1^2 + F_2^2\overline{\phi}_2^2 + F_3^2\overline{\phi}_3^2\right) \\ & + X_2''\left(q_1\overline{\phi}_1F_1 + q_2\overline{\phi}_2F_2 + q_3\overline{\phi}_3F_3\right)^2\big] \\ & + \Big[X_5F_i\overline{\phi}_i + X_5'\sum q_i^2F_i\overline{\phi}_i + X_6\overline{y}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3\Big]\overline{\lambda}\overline{\lambda}\Big\} \\ & + X_7\left(q_1^2\overline{\phi}_1\psi_1 + q_2^2\overline{\phi}_2\psi_2 + q_3^2\overline{\phi}_3\psi_3\right) \\ & \quad\times\left(q_1\overline{\phi}_1C\psi_1 + q_2\overline{\phi}_2C\psi_2 + q_3\overline{\phi}_3C\psi_3\right)\Big]. \end{aligned} \tag{7.26}$$

	X_1	$X_{2a,b,c}$	$X_{3a,b,c}$	X_2'
a	$6k_2y\overline{y}$			
b		$8k_2y\overline{y}$		$4k_2y\overline{y}$
c		$4k_1y\overline{y}$		$2k_1y\overline{y}$
d		$8k_3y\overline{y}$		$2k_3y\overline{y}$
e			$12k_3y\overline{y}$	
f			$6k_2y\overline{y}$	
g			$8k_4y\overline{y}$	

Table 7.3: Divergent contributions from Fig. 7.5

The diagrams contributing to X_1, etc. above are depicted in Figs. 7.5-7.10 whilst their divergent contributions are given individually in Tables 7.3-7.8 and in total by

$$\begin{aligned}
X_1^{(1)} &= \left(6k_2 - 6g^2\right) y\overline{y}L,\\
X_{2a}^{(1)} &= \left\{4\left(k_1 + 2k_2 + 2k_3\right) y\overline{y} + 2\left(1+\alpha\right) k_2 q_1 q_2 g^2\right\} L,\\
X_{2b}^{(1)} &= \left\{4\left(k_1 + 2k_2 + 2k_3\right) y\overline{y} + 2\left(1+\alpha\right) k_2 q_2 q_3 g^2\right\} L,\\
X_{2c}^{(1)} &= \left\{4\left(k_1 + 2k_2 + 2k_3\right) y\overline{y} + 2\left(1+\alpha\right) k_2 q_3 q_1 g^2\right\} L,\\
X_{3a}^{(1)} &= \left\{2\left(3k_2 + 6k_3 + 4k_4\right) y\overline{y} + \left(1+\alpha\right)\left[2\left(k_1 + 2k_2\right) q_2 q_3 - Qk_3\right] g^2\right\} L,\\
X_{3b}^{(1)} &= \left\{2\left(3k_2 + 6k_3 + 4k_4\right) y\overline{y} + \left(1+\alpha\right)\left[2\left(k_1 + 2k_2\right) q_3 q_1 - Qk_3\right] g^2\right\} L,\\
X_{3c}^{(1)} &= \left\{2\left(3k_2 + 6k_3 + 4k_4\right) y\overline{y} + \left(1+\alpha\right)\left[2\left(k_1 + 2k_2\right) q_1 q_2 - Qk_3\right] g^2\right\} L,\\
X_4^{(1)} &= -\left(1+\alpha\right)\left(k_2 + 2k_3 + 2k_4\right) Q g^2 L,\\
X_2'^{(1)} &= 2\left(k_1 + 2k_2 + k_3\right) y\overline{y}L,\\
X_2''^{(1)} &= -\frac{1}{4}\left(1+\alpha\right) g^4 \overline{y} L,\\
X_5^{(1)} &= \left[\left(4K_1 + 2K_2\right) y\overline{y} - g^2 y\overline{y}\right] L,\\
X_{5'}^{(1)} &= g^2\left(8K_1 - 10g^2\right) L,\\
X_6^{(1)} &= \left[2\left(7-\alpha\right) K_1 + \left(7-\alpha\right) K_2 + 14g^2\right] Q g^2 L,\\
X_7^{(1)} &= 16 g^4 L.
\end{aligned} \tag{7.27}$$

The terms involving X_2', X_2'' and X_5' are not contained in the original action whilst the term involving X_7 is not $\mathcal{N} = \frac{1}{2}$-invariant. However, we shall see later when we consider the non-linear renormalisation of auxiliary fields and gauginos that all of these terms may be removed, at least at one loop, by field redefinitions. The remaining diagrams which appear potentially to contribute divergences turn out to be zero or cancel; the diagram shown in Fig. 7.11 is zero by symmetry. The divergences from the diagrams of Fig. 7.12 are of the form

$$\overline{y}^{-1}\left[\left(q_2 - q_3\right) F_1 \psi_2 \left(C\psi_3\right) + \left(q_3 - q_1\right) F_2 \psi_3 \left(C\psi_1\right) + \left(q_1 - q_2\right) F_3 \psi_1 \left(C\psi_2\right)\right], \tag{7.28}$$

which is also not $\mathcal{N} = \frac{1}{2}$-invariant and which cannot be removed by a field redefinition. However, these diagrams also cancel.

Whilst non-linear renormalisation of the auxiliary fields was included in the last chapter merely for completeness, it is a necessity in the present case of a gauged model

	X_{2a}	X_{2b}	X_{2c}	X_{3a}	X_{3b}	X_{3c}	X_4
a	$2\alpha k_2 q_1 q_2 g^2$	$2\alpha k_2 q_2 q_3 g^2$	$2\alpha k_2 q_3 q_1$				
b	$2k_2 q_1 q_2 g^2$	$2k_2 q_2 q_3 g^2$	$2k_2 q_3 q_1$				
c				$-\alpha k_3 Q g^2$	$-\alpha k_3 Q g^2$	$-\alpha k_3 Q g^2$	
d				$-k_3 Q g^2$	$-k_3 Q g^2$	$-k_3 Q g^2$	
e				$2\alpha k_1 q_2 q_3 g^2$	$2\alpha k_1 q_3 q_1 g^2$	$2\alpha k_1 q_1 q_2 g^2$	
f				$2k_1 q_2 q_3 g^2$	$2k_1 q_3 q_1 g^2$	$2k_1 q_1 q_2 g^2$	
g				$4\alpha k_2 q_2 q_3 g^2$	$4\alpha k_2 q_3 q_1 g^2$	$4\alpha k_2 q_1 q_2 g^2$	
h				$4k_2 q_2 q_3 g^2$	$4k_2 q_3 q_1 g^2$	$4k_2 q_1 q_2 g^2$	
i							$-2\alpha k_4 Q g^2$
j							$-2k_4 Q g^2$
k							$-\alpha k_2 Q g^2$
l							$-k_2 Q g^2$
m							$-\alpha 2k_3 Q g^2$
n							$-2k_3 Q g^2$

Table 7.4: Divergent contributions from Fig. 7.6

	X_1	X_2''
a	$-6y\bar{y}g^2$	
b		$-\frac{1}{4}\alpha g^4$
c		$-\frac{1}{4}g^4$
d		0

Table 7.5: Divergent contributions from Fig. 7.7

	X_5'	X_5	X_6
a	$8g^2K_1$		
b		$4K_1y\bar{y}$	
c		$2K_2y\bar{y}$	
d			$-2\alpha Qg^2K_1$
e			$-2Qg^2K_1$
f			$16Qg^2K_1$
g			$8Qg^2K_1$
h			$-\alpha Qg^2K_2$
i			$-Qg^2K_2$

Table 7.6: Divergent contributions from Fig. 7.8

	X_5	X_5'	X_6
a	$-g^2y\bar{y}$		
b		$-8g^4$	
c		$-2g^4$	
d		0	
e			$8g^4Q$
f			$\frac{1}{2}\alpha Qg^4$
g			$\frac{1}{2}Qg^4$
h			$\frac{1}{2}(3+\alpha)Qg^4$
i			$4Qg^4$
j			$-\alpha Qg^4$
k			0

Table 7.7: Divergent contributions from Fig. 7.9

	X_7
a	$-4\alpha g^4$
b	$4(3+\alpha)g^4$
c	$-4\alpha g^4$
d	$4\alpha g^4$
e	$4g^4$

Table 7.8: Divergent contributions from Fig. 7.10

in the uneliminated formalism as was discussed in chapter 4. Furthermore, as our model is in non-anticommutative superspace, we also require a non-linear renormalisation of the gaugino field. In particular, we find it necessary at one loop to make the redefinitions

$$
\begin{aligned}
F_{1B}^{(1)} &= Z_F^{\frac{1}{2}(1)} F_1 - (\alpha+3)\, q_1^2 g^2 \overline{y} L \overline{\phi}_2 \overline{\phi}_3, \\
\overline{F}_{1B}^{(1)} &= Z_F^{\frac{1}{2}(1)} \overline{F}_1 - (\alpha+3)\, q_1^2 g^2 y L \phi_2 \phi_3 + (\alpha+9)\, i g^2 q_1^2 g L C^{\mu\nu} F_{\mu\nu} \overline{\phi}_1 \\
&\quad + k_1 g^2 L \Bigg[\frac{1}{2} (\alpha+3) \left(q_3^2 F_2 \overline{\phi}_1 \overline{\phi}_2 + q_2^2 F_3 \overline{\phi}_1 \overline{\phi}_3 \right) \\
&\quad + \alpha \overline{y} \left(q_1^2 - \frac{1}{2} q_2^2 - \frac{1}{2} q_3^2 \right) \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \\
&\quad + \overline{y} \left(q_1^2 + \frac{1}{2} q_2^2 + \frac{1}{2} q_3^2 \right) \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \Bigg] \\
&\quad + k_2 g^2 L \Bigg[\frac{1}{2} \alpha \left(q_3^2 F_2 \overline{\phi}_1 \overline{\phi}_2 + q_2^2 F_3 \overline{\phi}_1 \overline{\phi}_3 \right) + \alpha \overline{y} \left(2 q_1^2 - \frac{1}{2} q_2^2 - \frac{1}{2} q_3^2 \right) \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \\
&\quad - \left(q_1^2 + q_2^2 - \frac{1}{2} q_3^2 \right) F_2 \overline{\phi}_1 \overline{\phi}_2 - \left(q_1^2 + q_3^2 - \frac{1}{2} q_2^2 \right) F_3 \overline{\phi}_1 \overline{\phi}_3 \\
&\quad + \frac{1}{2} \overline{y} \left(q_2^2 + q_3^2 \right) \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \Bigg] \\
&\quad + k_3 g^2 \overline{y} L \left[\alpha q_1^2 - \left(3 q_1^2 + 2 q_2^2 + w q_3^2 \right) \right] \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \\
&\quad + 2 (k_1 + 2k_2 + k_3)\, y \overline{y} L F_1 \overline{\phi}_1^2 \\
&\quad - \frac{1}{4} (1+\alpha)\, g^4 q_1 \overline{\phi}_1 \left(q_1 F_1 \overline{\phi}_1 + q_2 F_2 \overline{\phi}_2 + q_3 F_3 \overline{\phi}_3 \right) \\
&\quad + \left[-10 g^2 + (7+\alpha) K_1 \right] g^2 L q_i^2 \overline{\phi} i \overline{\lambda} \overline{\lambda} \\
&\quad - \frac{1}{3} Q g^2 L \Bigg[2 \overline{y}^{-1} k_1 F_2 F_3 + k_1 \left(F_2 \overline{\phi}_1 \overline{\phi}_2 + F_3 \overline{\phi}_1 \overline{\phi}_3 \right) \\
&\quad + (2k_1 - 6k_3)\, \overline{y} \overline{\phi}_1^2 \overline{\phi}_2 \overline{\phi}_3 \Bigg] \\
&\quad + \overline{y}^{-1} \left[R^{(1)} F_2 F_3 + S^{(1)} \left(F_2 \overline{G}_3 + F_3 \overline{G}_2 \right) + T^{(1)} \overline{G}_2 \overline{G}_3 \right]
\end{aligned}
\tag{7.29}
$$

with similar expressions for $F_{2B,3B}^{(1)}$, $\overline{F}_{2B,3B}^{(1)}$, along with

$$
\lambda_B^{(1)} = Z_\lambda^{\frac{1}{2}(1)} \lambda + i\sqrt{2} g \sum_i \rho_i^{(1)} \overline{\phi}_i \left(C \psi_i \right). \tag{7.30}
$$

For the renormalisation constants at one loop, we have

$$
\begin{aligned}
Z_\lambda^{(1)} &= -2 g^2 L Q, \\
Z_A^{(1)} &= -2 g^2 L Q, \\
Z_g^{(1)} &= g^2 L Q, \\
Z_F^{(1)} &= -2 L y \overline{y}, \\
Z_{\phi_i}^{(1)} &= 2L \left[-y\overline{y} + (1-\alpha)\, g^2 q_i^2 \right], \qquad i = 1,2,3, \\
Z_{\psi_i}^{(1)} &= 2L \left[-y\overline{y} - (1+\alpha)\, g^2 q_i^2 \right], \qquad i = 1,2,3.
\end{aligned}
\tag{7.31}
$$

The ρ_i in the bare action, arising from the bare form of the gaugino field (Eq. (7.30)), produces the terms

$$\sum_i \rho_i g \left[\sqrt{2} C^{\mu\nu} \left(D_\mu \overline{\phi}_i \overline{\lambda} \overline{\sigma}_\nu \psi_i + \overline{\phi}_i \overline{\lambda} \overline{\sigma}_\nu D_\mu \psi_i\right) + 2\overline{\phi}_i \psi_i \left(\sum q_j \overline{\phi}_j C \psi_j\right)\right]. \tag{7.32}$$

Like the γ_i's in Eq. (7.2), these ρ_i's are purely divergent terms and at one loop we find it necessary to take

$$\begin{aligned} \gamma_i^{(1)} &= \left(8g^2 q_i^2 - 2y\overline{y}\right) L, \\ \rho_i^{(1)} &= 8g^2 q_i^2 L. \end{aligned} \tag{7.33}$$

With this value for ρ_i, the $\mathcal{N} = \frac{1}{2}$ non-invariant terms involving X_7 in Eq. (7.27) are cancelled at one loop.

With the bare auxiliary fields as given in Eq. (7.29), the renormalisation of the Yukawa couplings is found to be as we would expect from applying the non-renormalisation theorem in the superfield formalism, namely

$$\begin{aligned} y_B &= \mu^{\frac{1}{2}\epsilon} Z_{\Phi_1}^{-\frac{1}{2}} Z_{\Phi_2}^{-\frac{1}{2}} Z_{\Phi_3}^{-\frac{1}{2}} y, \\ \overline{y}_B &= \mu^{\frac{1}{2}\epsilon} Z_{\Phi_1}^{-\frac{1}{2}} Z_{\Phi_2}^{-\frac{1}{2}} Z_{\Phi_3}^{-\frac{1}{2}} \overline{y}, \end{aligned} \tag{7.34}$$

where Z_{Φ_i}, $i = 1, 2, 3$ are the renormalisation constants for the chiral superfields as computed in a supersymmetric gauge. At one loop these are given by

$$Z_{\Phi_i}^{(1)} = 2L\left[-y\overline{y} + 2g^2 q_i^2\right], \qquad i = 1, 2, 3. \tag{7.35}$$

From Eqs. (7.34), (7.35), the β-function for y is found to be given by

$$\beta_y^{(1)} = \frac{1}{16\pi^2}\left(3y\overline{y} - 2g^2 Q\right) y, \tag{7.36}$$

with a similar expression for $\beta_{\overline{y}}^{(1)}$

Note that if we set $q_1 = -q_2 = q$ and $y = \overline{y} = \sqrt{2} g q$ then Eq. (7.36) reduces to

$$\beta_g^{(1)} = 2q^2 \frac{g^3}{16\pi^2}, \tag{7.37}$$

which is in agreement with the one-loop gauge β-function.

The bare couplings may be calculated from Eqs. (7.15), (7.27), (7.29), (7.31), (7.33), (7.34) and are given by

$$\begin{aligned} k_{1B}^{(1)} &= 6\left(k_1 + k_2 - g^2\right) y\overline{y} L - 3R^{(1)}, \\ k_{2B}^{(1)} &= 4\left(k_1 + 3k_2 + 2k_3\right) y\overline{y} L + R^{(1)} - S^{(1)}, \\ k_{3B}^{(1)} &= 2\left(k_1 + 5k_2 + 8k_3 + 4k_4\right) y\overline{y} L + S^{(1)} - T^{(1)}, \\ k_{4B}^{(1)} &= 3T^{(1)}, \\ K_{1B}^{(1)} &= \left(\left[6K_1 + 2K_2\right] y\overline{y} + 2Qg^2 K_1 - g^2 y\overline{y}\right) L, \\ K_{2B}^{(1)} &= 2\left(12K_1 + 5K_2 + 2g^2\right) Qg^2 L, \end{aligned} \tag{7.38}$$

where R, S and T were introduced in Eq. (7.29) to represent possible additional renormalisations of F_i which are not determined by the requirements of renormalisability.

The renormalisation of $\overline{F}_i$ as given in Eq. (7.29) is largely determined by the requirement that the couplings k_{1-4}, $K_{1,2}$ are multiplicatively renormalised as described above. However, we still have the freedom to choose the parameters $R^{(1)}$, $S^{(1)}$, $T^{(1)}$ (which are the same for each $\overline{F}_{iB}$). Choosing $R^{(1)} = S^{(1)} = T^{(1)} = 0$ in Eq. (7.29) leaves almost the minimal renormalisation of $\overline{F}_i$ possible to ensure multiplicative renormalisation. However, we have included the terms with a factor Q in Eq. (7.29) in order to remove $g^2 k_i$-dependent terms in k_{1-4B}.

Requiring that k_{iB} and $K_{1,2B}$ be independent of μ, we find that the β-functions for these constants are given by

$$\begin{aligned}
\beta_{k_1}^{(1)} &= \frac{1}{16\pi^2}\left[6\left(k_1 + k_2 - g^2\right) y\overline{y} - 3r\right], \\
\beta_{k_2}^{(1)} &= \frac{1}{16\pi^2}\left[4\left(k_1 + 3k_2 + 2k_3\right) y\overline{y} + r - s\right], \\
\beta_{k_3}^{(1)} &= \frac{1}{16\pi^2}\left[2\left(k_1 + 5k_2 + 8k_3 + 4k_4\right) y\overline{y} + s - t\right], \\
\beta_{k_4}^{(1)} &= \frac{1}{16\pi^2} \times 3t, \\
\beta_{K_1}^{(1)} &= \frac{1}{16\pi^2}\left[\left(6K_1 + 2K_2\right) y\overline{y} + 2Qg^2 K_1 - g^2 y\overline{y}\right], \\
\beta_{K_2}^{(1)} &= \frac{1}{16\pi^2} 2\left(12K_1 + 5K_2 + 2g^2\right) Qg^2,
\end{aligned} \tag{7.39}$$

with $R^{(1)} = rL$, etc. Note that these β-functions are different from those derived in the ungauged case in the previous chapter. Our three-field superpotential is, of course, also somewhat different from that used in the ungauged case. Also, we have had to include $F_i\overline{\phi}_i^2$ terms as non-linear terms in $\overline{F}_{iB}$ in order to remove the X_2' terms which would have spoiled renormalisability and these have contributed to k_{3B}. It seems impossible to use the freedom to choose $R^{(1)}$, $S^{(1)}$, $T^{(1)}$ in Eq. (7.29) to make the two sets of β-functions agree.

7.2.2 Renormalisation in the Eliminated Case

Eliminating the auxiliary fields F_i and $\overline{F}_i$ from the action we find

$$\begin{aligned}
F_i &= \overline{G}_i, \\
\overline{F}_1 &= G_1 - \overline{y}^{-1}|C|^2\left[k_1 F_2 F_3 + k_2\left(F_2\overline{G}_3 + F_3\overline{G}_2\right) + k_3\overline{G}_2\overline{G}_3\right] \\
&\quad - igC^{\mu\nu}F_{\mu\nu}\overline{\phi}_1 - \frac{1}{4}g^2|C|^2 K_1\overline{\phi}_1\overline{\lambda}\overline{\lambda},
\end{aligned} \tag{7.40}$$

with similar expressions for $\overline{F}_2$ and $\overline{F}_3$. The action then becomes

$$\begin{aligned}
S = \int d^4x\Big[&-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\overline{\lambda}\overline{\sigma}^{\mu}\left(D_\mu\lambda\right) + \frac{1}{2}D^2 - igC^{\mu\nu}F_{\mu\nu}\overline{\lambda}\overline{\lambda} \\
&- i\overline{\psi}_i\overline{\sigma}^{\mu}\left(D_\mu\psi_i\right) - \left(D^\mu\overline{\phi}\right)_i - \left(D^\mu\overline{\phi}\right)_i\left(D_\mu\phi\right)_i \\
&+ g\sum\Big\{q_i\overline{\phi}_i D\phi + i\sqrt{2}gq_i\left(\overline{\phi}_i\lambda\psi_i - \overline{\psi}_i\overline{\lambda}\phi_i\right) \\
&- \gamma_i C^{\mu\nu}g\left(\sqrt{2}D_\mu\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu\psi_i + \sqrt{2s}\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu D_\mu\psi_i\right)\Big\} \\
&+ \sqrt{2}gC^{\mu\nu}D_\mu\overline{\phi}_i\overline{\lambda}\overline{\sigma}_\nu\psi_i - G_i\overline{G}_i
\end{aligned} \tag{7.41}$$

$$
\begin{aligned}
&+ y\left(\phi_1\psi_2\psi_3 + \phi_2\psi_3\psi_1 + \phi_3\psi_1\psi_2\right) \\
&+ \overline{y}\left(\overline{\phi}_1\overline{\psi}_2\overline{\psi}_3 + \overline{\phi}_2\overline{\psi}_3\overline{\psi}_1 + \overline{\phi}_3\overline{\psi}_1\overline{\psi}_2\right) \\
&+ ig\overline{y}\left(1 - \gamma_1 - \gamma_2 - \gamma_3\right) C^{\mu\nu} F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3 \\
&+ \left.\lambda_1\overline{y}^{-1}|C|^2\overline{G}^3 + \lambda_2\overline{y}|C|^2\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3\overline{\lambda}\overline{\lambda}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1 &= k_1 + 3\left(k_2 + k_3\right) + k_4, \\
\lambda_2 &= 3K_1 + K_2.
\end{aligned} \tag{7.42}
$$

The renormalisation of the last three terms in Eq. (7.41) now needs to be reconsidered. First, we consider the $C^{\mu\nu}F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3$ term. Its coefficient has changed and in particular we see, comparing Eqs. (7.4), (7.41), that its finite part (that containing no γ_i) has changed by a factor of $-\frac{1}{2}$ and the diagrams (Figs. 7.4(e)-(h)) which cancelled the contributions from Figs. 7.4(i)-(l) are no longer present whilst these latter contributions are multiplied by $-\frac{1}{2}$. Moreover, as the eliminated theory in Eq. (7.41) also contains a $\overline{G}_iG_i$ vertex which was not present in the uneliminated case, there is a new diagram as depicted in Fig. 7.13 giving a divergent contribution

$$
-6iy\overline{y}^2C^{\mu\nu}\int \mathrm{d}^4x F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3. \tag{7.43}
$$

However, taking all these changes into account, we find that the divergences are still cancelled as they should be.

The remaining two terms need to be examined in more detail. We write the divergent contributions to these terms as

$$
\Gamma^{\text{pole}}_{C\ \text{elim}} = -|C|^2\int \mathrm{d}^4x\left[Y_1\overline{y}^{-1}\overline{G}_1\overline{G}_2\overline{G}_3 + Y_2\overline{y}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3\overline{\lambda}\overline{\lambda}\right]. \tag{7.44}
$$

Most of the relevant contributions to Y_1 can be read off from those to X_4 in Table 7.4(i),(j) with k_4 replaced here by λ_1. Similarly, most of the relevant contributions to Y_2 can be read off from those to X_6 in Tables 7.6(g)-(i), 7.6(e)-(k) with K_2 replaced by λ_2. However, in the eliminated case, there are also diagrams with a $g\overline{y}C^{\mu\nu}F_{\mu\nu}\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3$ vertex. Such diagrams were previously cancelled by diagrams with an internal F propagator in a similar fashion to Figs. 7.4(e)-(h) and Figs. 7.4(i)-(l) but no such diagrams are present in the eliminated case. Again, there are further diagrams incorporating the $\overline{G}_iG_i$ vertex which was not present in the uneliminated case. With these two additional cases in mind we now need to include contributions from the diagrams shown in Fig. 7.14. The contributions from Figs. 7.14(a)-(i) are listed in Table 7.9 whilst those from Figs. 7.14(j),(k) cancel and are omitted.

We find from the eliminated diagrams that

$$
\begin{aligned}
Y_1^{(1)} &= 2\left[12y\overline{y}\lambda_1 - (1+\alpha)\,g^2Q\lambda_1 - 3g^2y\overline{y}\right]L, \\
Y_2^{(1)} &= \left[6y\overline{y}\lambda_2 + (7-\alpha)\,Qg^2\lambda_2 + 4Qg^4 - 3g^2y\overline{y}\right]L,
\end{aligned} \tag{7.45}
$$

and

$$
\begin{aligned}
\beta^{(1)}_{\lambda_1} &= \frac{1}{16\pi^2}\left(24\lambda_1y\overline{y} - 6g^2y\overline{y}\right), \\
\beta^{(1)}_{\lambda_2} &= \frac{1}{16\pi^2}\left(6y\overline{y}\lambda_2 + 10Qg^2\lambda_2 + 4Qg^4 - 3g^2y\overline{y}\right).
\end{aligned} \tag{7.46}
$$

	Y_1	Y_2
a	$24y\bar{y}\lambda_1$	
b	$-6g^2y\bar{y}$	
c	0	
d	0	
e		$6y\bar{y}\lambda_2$
f		$-8Qg^4$
g		$-2Qg^4$
h		0
i		$-3g^2y\bar{y}$

Table 7.9: Divergent contributions from Fig. 7.14

Checking again for consistency between formalisms as in the ungauged case, we find using Eqs. (7.38), (7.45) that

$$\begin{aligned}\lambda_{1B} &= k_{1B} + 3\left(k_{2B} + k_{3B}\right) + k_{4B},\\ \lambda_{2B} &= 3K_{1B} + K_{2B}\end{aligned} \tag{7.47}$$

is satisfied at one loop. Our ability to remove g^2k_i terms from $k_{iB}^{(1)}$ in the uneliminated case is now seen to follow from the fact that, in the eliminated case, $\lambda_{1B}^{(1)}$ contains no $g^2\lambda_1$ terms. The non-gauge part of this component calculation is in agreement with the corresponding part of the superfield calculation of Ref. [96]. Whilst our calculation has three chiral fields with charges of q, $-q$ and 0, the calculation in Ref. [96] was performed in the adjoint representation of $U(N)$ in the limit $N \to 1$.

7.2.3 A Natural Form for the Action

The classical deformed Wess-Zumino action of Eq. (7.1) corresponded to the values $k_1 = y$, $K_1 = \frac{1}{4}$, $k_{2-4} = K_2 = 0$. However, the more general renormalisable action in Eq. (7.15) is $\mathcal{N} = \frac{1}{2}$-invariant regardless of the values of k_{1-4}, $K_{1,2}$ and we see from Eq. (7.39) that the choice derived from the classical action is not maintained under renormalisation: if we set the couplings to the above values at one scale, different values are inevitably generated at other scales. In the previous chapter, we looked for a natural set of values for k_{1-4} in the ungauged case which *is* preserved under renormalisation.

If we ask the same question here, we find that the values of k_{1-4} and ρ must satisfy the sole condition

$$\left[(24 - 6\rho)\, y\bar{y} + 4\rho Q g^2\right]\lambda_1 = 6g^2y\bar{y}, \tag{7.48}$$

which is the same condition we would find in the eliminated case using Eq. (7.46). In the ungauged case we once again find that the the particular solutions

$$k_1 = -k_2 = k_3 = -k_4, \qquad \rho = 0 \tag{7.49}$$

and

$$k_1 = -\frac{3}{2}k_2 = 3k_3, \qquad k_4 = 0, \qquad \rho = \frac{1}{3} \tag{7.50}$$

require no non-linear renormalisation of F_i.

In the ungauged case of the previous chapter we found four sets of parameters that obeyed the naturalness conditions derived there to one loop and that this was reduced to two sets at two loops. We find that these two sets of parameters perform a similar function for the β-functions in the gauged case with $r = s = t = 0$ derived using the minimal renormalisation of the F_i consistent with renormalisability, despite the difference in β-functions between the two cases.

7.3 Summary

We have performed a full one-loop analysis of the renormalisation of the deformed $U(1)$-gauged Wess-Zumino model with a superpotential and have found that the results in the eliminated and uneliminated cases are again consistent provided we include a full set of terms generated by renormalisation. This is the first complete one-loop calculation of a general non-anticommutative supersymmetric gauge theory with a superpotential as $y\bar{y}$ contributions to S_{kin} were omitted in Ref. [54]. It was necessary, as discussed in §4.3, to include non-linear renormalisation of the gaugino as well as a renormalisation parametrised by γ_i in Eq. (7.2). These renormalisations were determined in this case through consideration of the theory with a superpotential and were found to contain y-independent terms that would not be required without a superpotential. Although this is an unusual case of affairs, it is in agreement with the y-independent parts of the renormalisation of the $U(1)$ part of the $SU(N) \otimes U(1)$ theory in [54] despite that calculation being performed in the presence of a mass term and the present case including a trilinear superpotential.

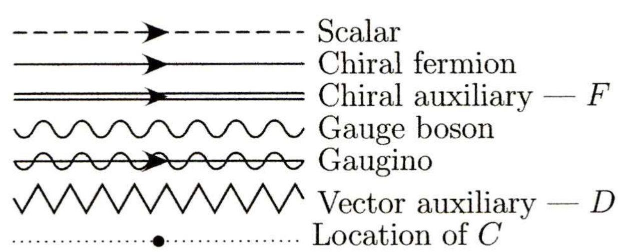


Figure 7.1: Conventions for Feynman diagrams in supersymmetric gauge theory

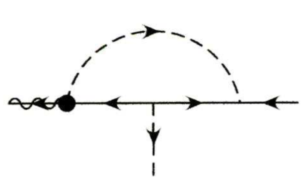


Figure 7.2: One-loop diagram with a C vertex and one gaugino, one ψ and one $\overline{\phi}$ external legs

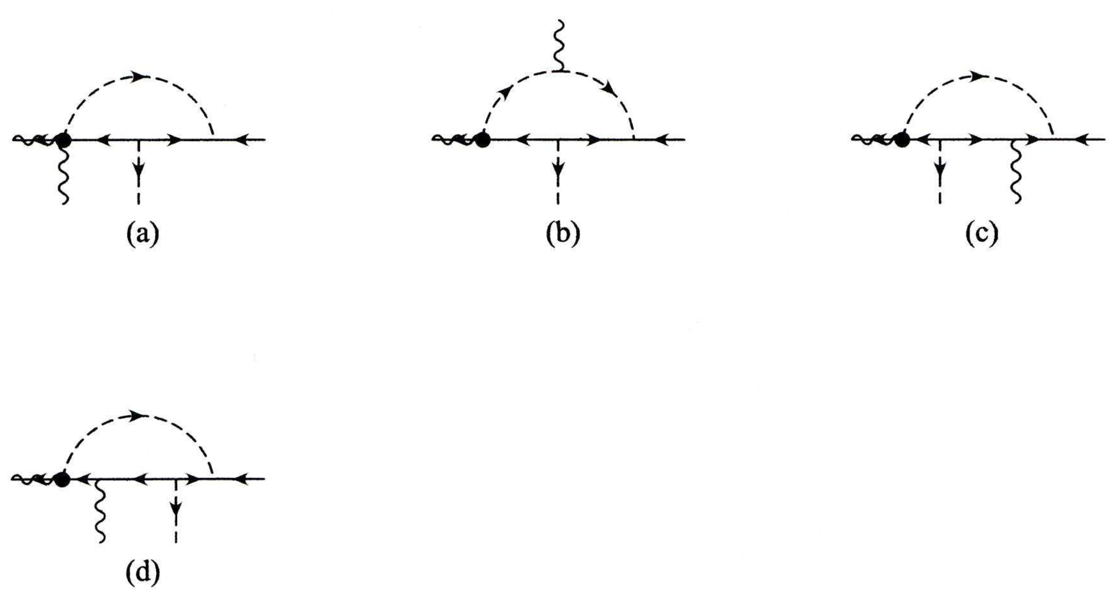


Figure 7.3: One-loop diagrams with a C vertex and one gauge, one gaugino, one ψ and one $\overline{\phi}$ external legs

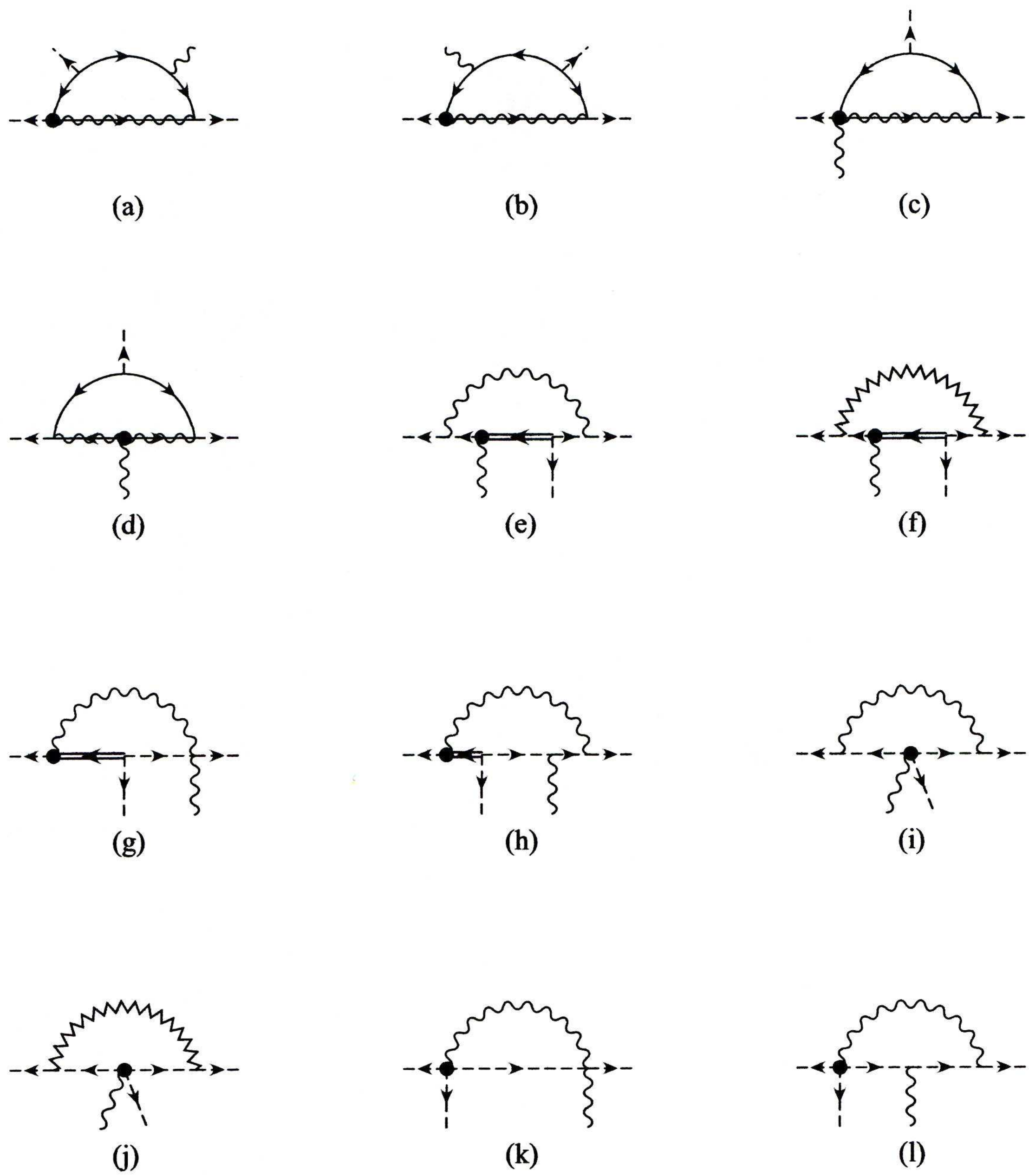


Figure 7.4: One-loop diagrams with a C vertex and three $\overline{\phi}$ and one gauge-field external legs

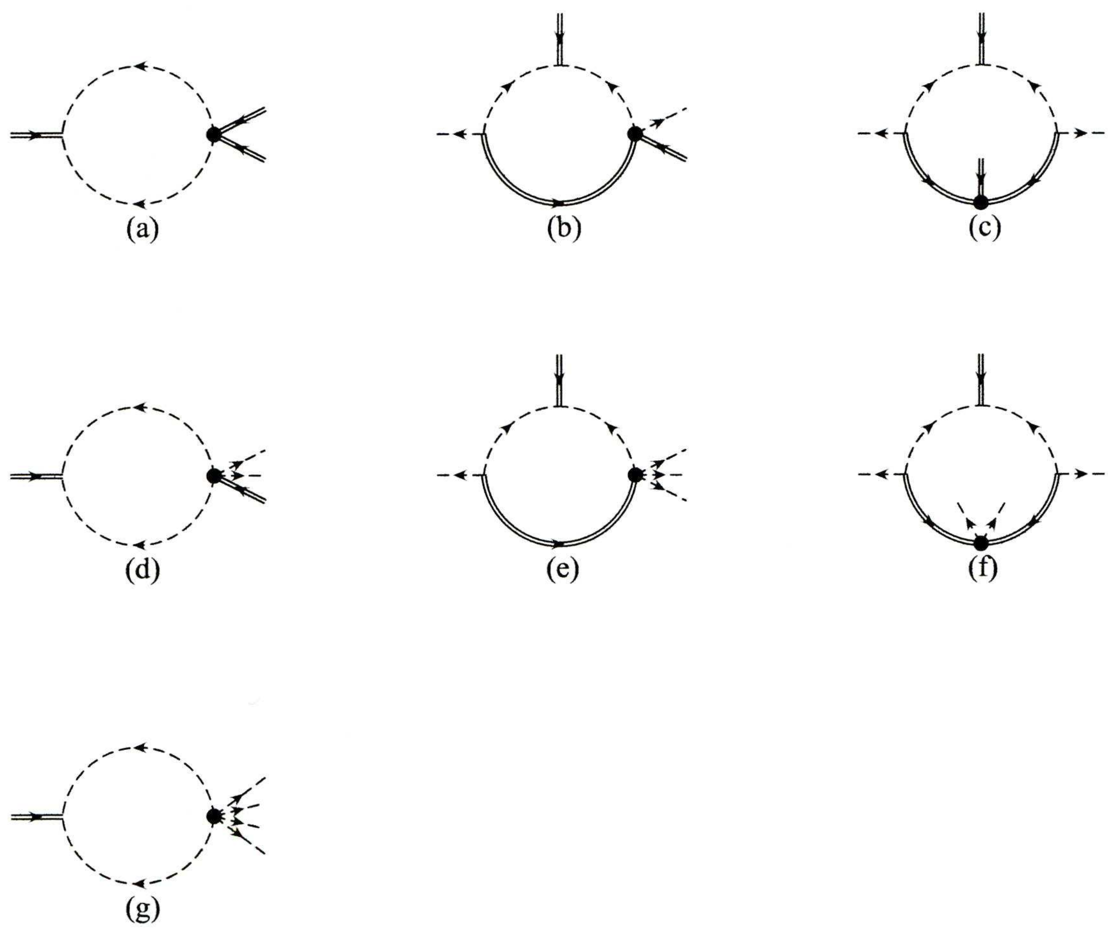


Figure 7.5: One-loop diagrams with a $|C|^2$ vertex, F or $\overline{\phi}$ external legs and purely F or $\overline{\phi}$ internal propagators

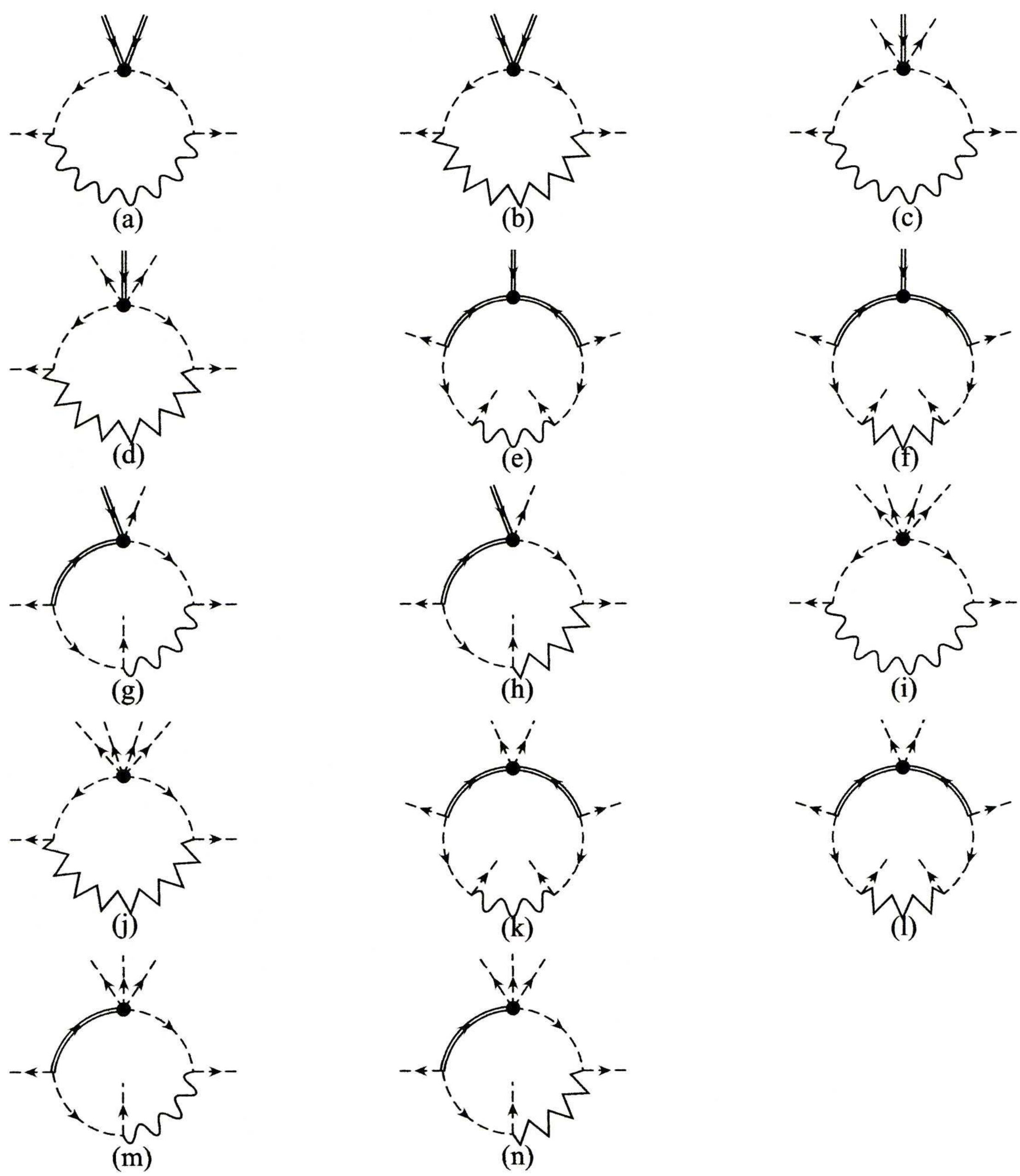


Figure 7.6: One-loop diagrams with a $|C|^2$ vertex, F or $\overline{\phi}$ external legs and an internal gauge or D propagator

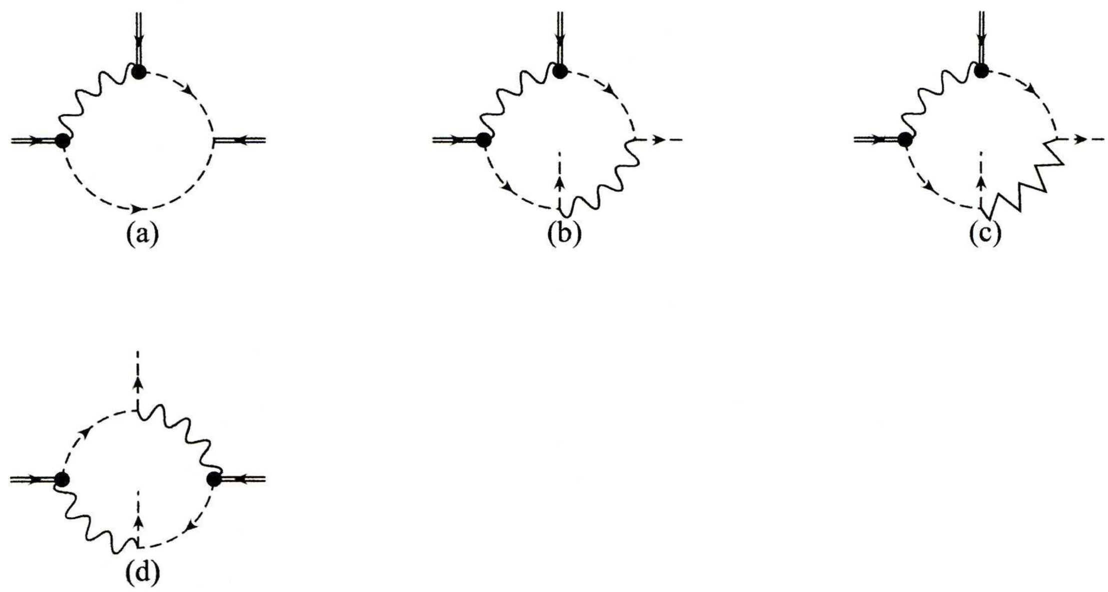


Figure 7.7: One-loop diagrams with two $C^{\mu\nu}$ vertices, F or $\overline{\phi}$ external legs and an internal gauge or D propagator

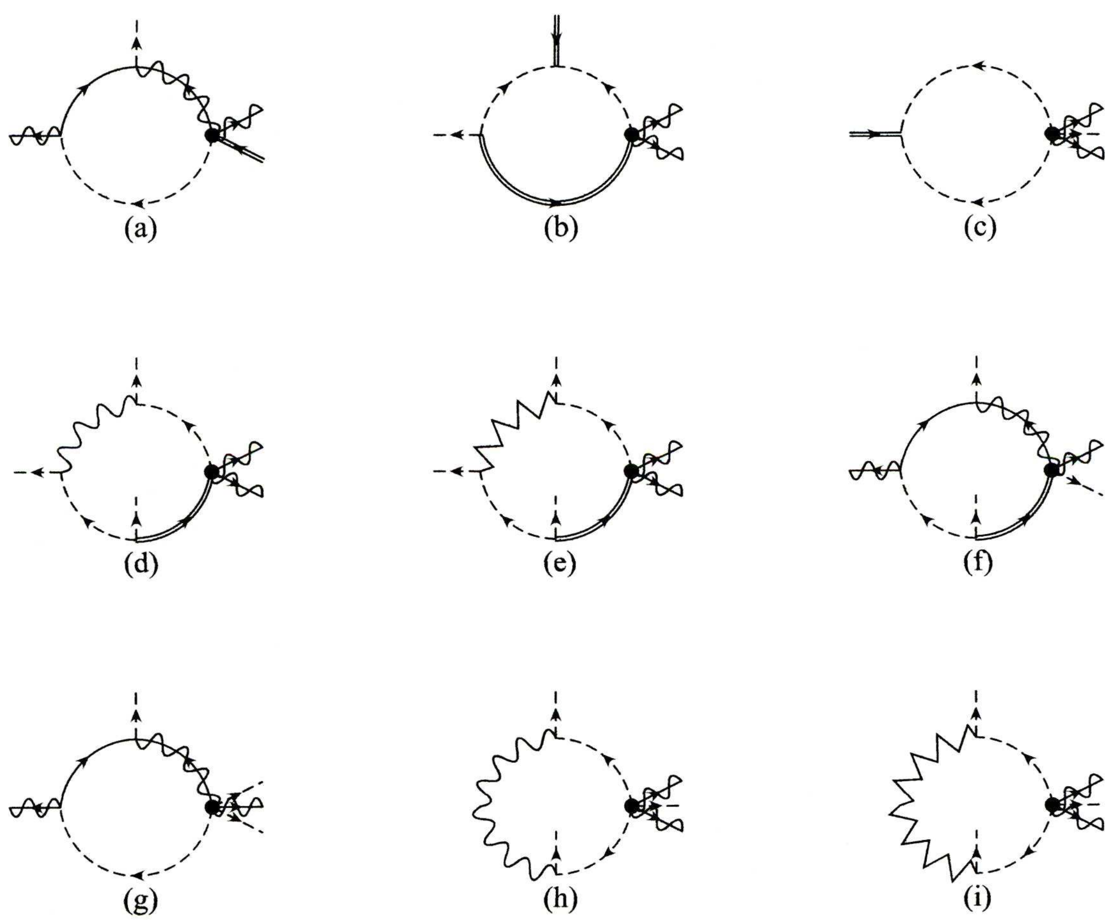


Figure 7.8: One-loop diagrams with a $|C|^2$ vertex, and two gaugino and F or $\overline{\phi}$ external legs

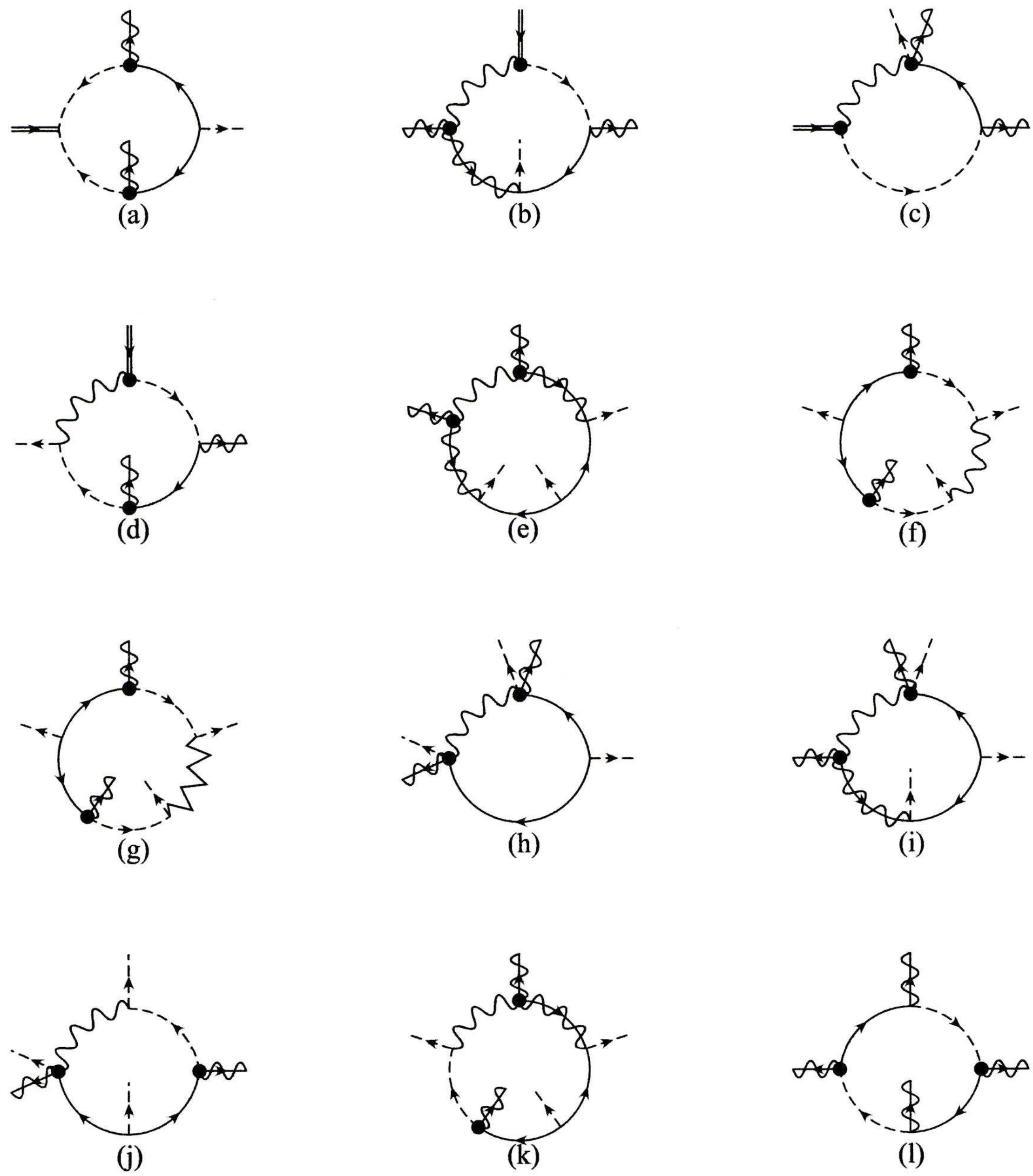
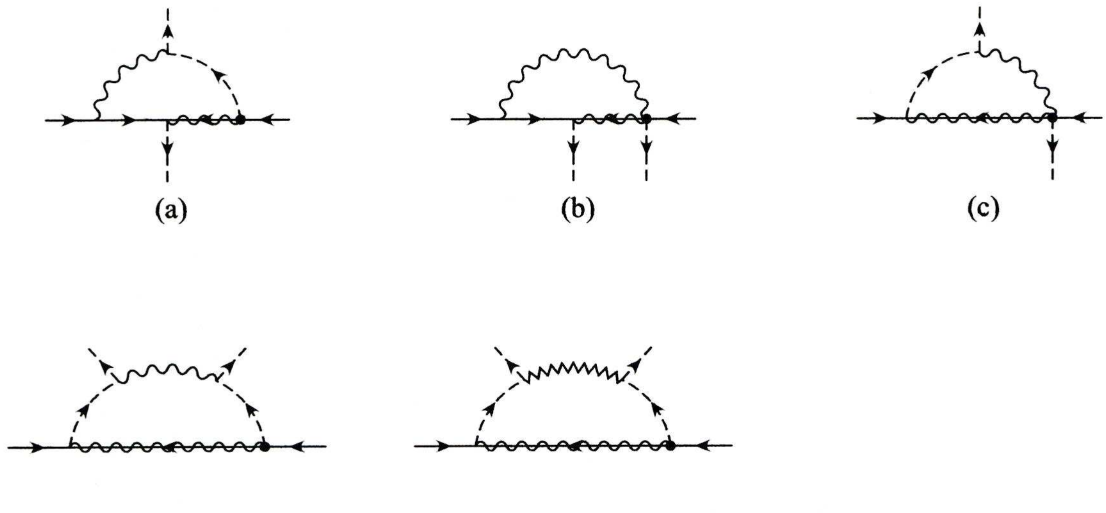


Figure 7.9: One-loop diagrams with two $C^{\mu\nu}$ vertices, and two gaugino and F or $\overline{\phi}$ external legs



(a) (b) (c)

(c) (d)

Figure 7.10: One-loop diagrams with two $\overline{\phi}$ and two ψ external legs (and no Yukawa vertices)

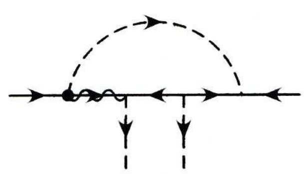


Figure 7.11: One-loop diagram with two $\overline{\phi}$ and two ψ external legs (and two Yukawa vertices)

(a) (b)

Figure 7.12: One-loop diagrams with one F and two ψ external legs

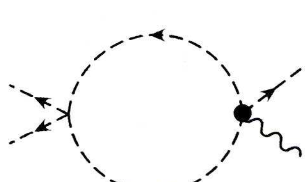


Figure 7.13: Additional one-loop diagram for the eliminated case

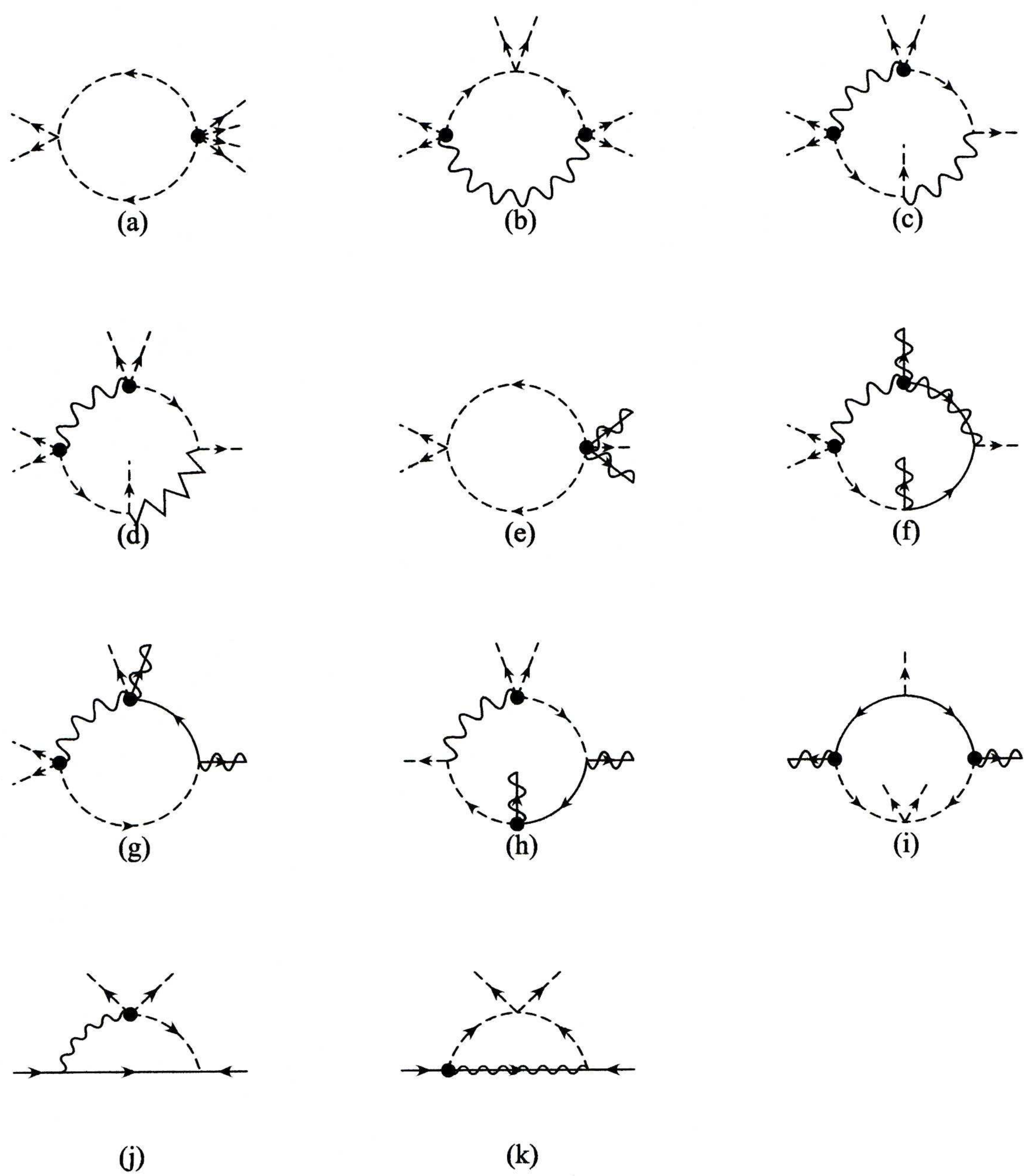


Figure 7.14: Further one-loop diagrams for the eliminated case

Chapter 8

Deformed Non-Abelian Gauge Theory

We now discuss a modification of non-anticommutative gauge theory which will allow for a renormalisable theory with a trilinear superpotential. We begin by discussing the problems faced when the introduction of such a superpotential is attempted and then we discuss the solution put forward in [97] and its reformulation in terms of component fields [98].

8.1 Restrictions on $\mathcal{N} = \frac{1}{2}$ theories

There are a number of apparent restrictions placed on non-anticommutative theories by the constraints of renormalisability and symmetry. Firstly it should be noted that the Wess-Zumino model remains renormalisable when extended to the non-anticommutative case as was discussed in chapter 6. Problems with renormalisability arise, however, when we consider non-anticommutative gauge theories. Considering first a pure gauge theory with no chiral matter, we find that $U(1)$ theories are renormalisable but that a non-Abelian $U(N)$ symmetry will not be preserved under renormalisation. In particular, form invariance is lost as a non-Abelian symmetry leads to the appearance of new terms in the renormalised action which do not appear in the classical action. This, however, is only due to the difference in renormalisation between the $U(1)$ and $SU(N)$ parts of the theory and, indeed, an $SU(N)$ theory is found to be form-invariant [99].

Upon the introduction of chiral fields coupled to the gauge fields, a new problem arises: now it appears that an $SU(N)$ symmetry is also not form-invariant. We could imagine introducing matter in various representations of the gauge symmetry group, the most natural examples being the fundamental/antifundamental representation and the adjoint representation, both of which we consider here. In the former case, $SU(N)$ gauge transformations of the non-anticommutative terms lead to new terms in the action that would be cancelled by the $U(1)$ transformations in the $U(N)$ case. However, as a $U(N)$ group is disallowed by $\mathcal{N} = \frac{1}{2}$ transformations, these cancellations can no longer take place. For this reason it is necessary to reintroduce the $U(1)$ symmetry with its own coupling to allow for gauge- and $\mathcal{N} = \frac{1}{2}$-invariance. In the case of matter in the adjoint representation of the gauge group, an $SU(N)$ group leads to a renormalisable theory only if there is no trilinear superpotential. If a trilinear term is introduced, $\mathcal{N} = \frac{1}{2}$-invariance links a number of terms from the $SU(N)$ and $U(1)$ parts of a $U(N)$ theory, leading us again to the conclusion that an $SU(N)$ symmetry cannot be maintained at the quantum level without the introduction of an independent $U(1)$ symmetry. A trilinear superpotential in the fundamental case is incompatible with $SU(N) \otimes U(1)$- and $\mathcal{N} = \frac{1}{2}$-invariance even at the classical level.

We are now equipped to appreciate the problem of introducing a trilnear superpotential. At the classical level, $\mathcal{N} = \frac{1}{2}$-invariance of Yukawa terms requires the chiral matter to be in the adjoint representation of $U(N)$ as $\mathcal{N} = \frac{1}{2}$ transformations mix $SU(N)$ and $U(1)$ fields. However, as the $SU(N)$ and $U(1)$ fields renormalise differently, renormalisabilty requires an $SU(N) \otimes U(1)$ gauge symmetry with separate couplings. This discrepancy apparently rules out the possibility of constructing a renormalisable non-Abelian gauge-invariant $\mathcal{N} = \frac{1}{2}$ theory with a superpotential which was precisely the motivation for our study of an Abelian model in the previous chapter. However, means of circumventing this limitation was realised in [97].

8.2 A Non-Abelian $\mathcal{N} = \frac{1}{2}$ Theory with a Superpotential

As the renormalisation of the $SU(N)$ Yukawa coupling is related to that of the $SU(N)$ superfields whilst the renormalisation of the $U(1)$ coupling is related to those of both the $SU(N)$ and $U(1)$ fields, an adjustment of the $U(1)$ superfield renormalisation will bring the renormalisation of the two Yukawa couplings back into agreement, allowing for a single coupling. This would require the introduction of a new coupling, κ, which multiplies the $U(1)$ chiral superfield's kinetic term. In order to achieve this without destroying the $\mathcal{N} = \frac{1}{2}$ invariance, κ must also multiply any terms related to the kinetic term by $\mathcal{N} = \frac{1}{2}$ transformations. κ may then be adjusted to allow for a unification of Yukawa couplings: this is the prescription used in [97] for a renormlisable non-Abelian theory. Whilst [97] worked in superfields, however, we wish to confirm the results in the uneliminated component formalism.

The classical action with a superpotential may be written

$$
\begin{aligned}
S_0 = \int d^4x \Big\{ & e^{AB} \left(-\frac{1}{4} F^{\mu\nu A} F^B_{\mu\nu} - i\overline{\lambda}^A \overline{\sigma}^\mu \left(D_\mu \lambda \right)^B + \frac{1}{2} D^A D^B \right) \\
& - \frac{1}{2} i C^{\mu\nu} d^{ABC} e^{AD} F^D_{\mu\nu} \overline{\lambda}^B \overline{\lambda}^C \\
& + \overline{F} F - i\overline{\psi}\overline{\sigma}^\mu D_\mu \psi - D^\mu \overline{\phi} D_\mu \phi + \overline{\phi} D^F \phi + i\sqrt{2} \left(\overline{\phi} \lambda^F \psi - \overline{\psi} \overline{\psi}^F \phi \right) \\
& + C^{\mu\nu} \left(\sqrt{2} D_\mu \overline{\phi} \overline{\lambda}^D \overline{\sigma}_\nu \psi + i\overline{\phi} F^D_{\mu\nu} F \right) \\
& + (\kappa - 1) \Big[\overline{F}^0 F^0 - i\overline{\psi}^0 \overline{\sigma}^\mu \partial_\mu \psi^0 - \partial^\mu \overline{\phi}^0 \partial_\mu \phi^0 \\
& + d^{000} C^{\mu\nu} \left(\sqrt{2} \partial_\mu \overline{\phi}^0 \overline{\lambda}^0 \overline{\sigma}_\nu \psi^0 + i\overline{\phi}^0 F^0_{\mu\nu} F^0 \right) \\
& + d^{ab0} C^{\mu\nu} \left(\sqrt{2} D_\mu \overline{\phi}^a \overline{\lambda}^b \overline{\sigma}_\nu \psi^0 + i\overline{\phi}^a F^b_{\mu\nu} F^0 \right) \Big] \\
& + \frac{1}{2} \Big(y d^{ABC} \phi^A \phi^B F^C - y d^{ABC} \phi^A \psi^B \psi^C \\
& + \overline{y} d^{ABC} \overline{\phi}^A \overline{\phi}^B \overline{F}^C - \overline{y} d^{ABC} \overline{\phi}^A \overline{\psi}^B \overline{\psi}^C \Big) \\
& + \frac{1}{3} i\overline{y} C^{\mu\nu} f^{abc} D_\mu \overline{\phi}^a D_\nu \overline{\phi}^b \overline{\phi}^c - \frac{1}{3} i\overline{y} C^{\mu\nu} d^{ABE} d^{CDE} F^D_{\mu\nu} \overline{\phi}^A \overline{\phi}^B \overline{\phi}^C \\
& + \kappa_1 \sqrt{2} C^{\mu\nu} d^{abc} \left(\overline{\phi}^a \overline{\lambda}^b \overline{\sigma}_\nu D_\mu \psi^c + D_\mu \overline{\phi}^a \overline{\lambda}^b \overline{\sigma}_\nu \psi^c + i\overline{\phi}^a F^b_{\mu\nu} F^c \right)
\end{aligned}
\qquad (8.1)
$$

$$+ \kappa_2\sqrt{2}C^{\mu\nu}d^{ab0}\left(\overline{\phi}^0\overline{\lambda}^a\overline{\sigma}_\nu D_\mu\psi^b + \partial_\mu\overline{\phi}^0\overline{\lambda}^a\overline{\sigma}_\nu\psi^b + i\overline{\phi}^0 F^a_{\mu\nu}F^b\right)$$
$$+ \kappa_3\sqrt{2}C^{\mu\nu}d^{ab0}\left(\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\partial_\mu\psi^0 + D_\mu\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^0 + i\overline{\phi}^a F^b_{\mu\nu}F^0\right)$$
$$+ \kappa_4\sqrt{2}C^{\mu\nu}d^{0ab}\left(\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu D_\mu\psi^b + D_\mu\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu\psi^b + i\overline{\phi}^a F^0_{\mu\nu}F^b\right)$$
$$+ \kappa_5\sqrt{2}C^{\mu\nu}d^{000}\left(\overline{\phi}^0\overline{\lambda}^0\overline{\sigma}_\nu\partial_\mu\psi^0 + \partial_\mu\overline{\phi}^0\overline{\lambda}^0\overline{\sigma}_\nu\psi^0 + i\overline{\phi}^0 F^0_{\mu\nu}F^0\right)\Big\},$$

where, for convenience, we have used the following shorthand notations for group theory factors:

$$\begin{aligned} \lambda^F &= \lambda^A\widetilde{F}^A, & \left(\widetilde{F}^B\right)^{AC} &= if^{ABC}, \\ \lambda^D &= \lambda^A\widetilde{D}^A, & \left(\widetilde{D}^B\right)^{AC} &= d^{ABC}, \end{aligned} \tag{8.2}$$

where f^{ABC} and d^{ABC} are the antisymmetric and symmetric group structure constants and we have similar expressions for D^F, $F^D_{\mu\nu}$. The gauge-covariant derivative and field strength are given by

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + iA^F_\mu\phi, \\ F^A_{\mu\nu} &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC}A^B_\mu A^C_\nu, \end{aligned} \tag{8.3}$$

with similar definitions for $D_\mu\psi$ and $D_\mu\lambda$. Although our gauge group is $SU(N)\otimes U(1)$, we use a $U(N)$ notation in which capital letter indices represent $U(N)$ whilst lower-case indices represent the $SU(N)$ component and 0 the $U(1)$ component. Note that $f^{ABC}=0$ unless all indices are $SU(N)$. Also, we have

$$\begin{aligned} d^{ab0} &= \sqrt{\frac{2}{N}}\delta^{ab}, \\ d^{000} &= \sqrt{\frac{2}{N}}, \end{aligned} \tag{8.4}$$

and other useful indentities for U(N) are listed in Appendix B. For a more comprehensive treatment of such identities see, for example, Ref. [100].

To account for the different subgroups, the couplings e^{AB} are given by

$$e^{ab} = \frac{1}{g^2}\delta^{ab}, \qquad e^{00} = \frac{1}{g_0^2}, \qquad e^{0a} = e^{a0} = 0. \tag{8.5}$$

If the classical action is derived by reducing the superspace action to components, a number of terms appear which are individually $\mathcal{N}=\frac{1}{2}$-invariant such as terms involving only $\overline{\phi}$, $\overline{\lambda}$ or F. These terms are also generated by quantum corrections even if they are not present in the classical action. However, as such terms do not influence the renormalisability of the theory, which is our main concern, they are omitted here (though they were given a full treatment in Ref. [97]). On the other hand, we have included some additional sets of terms that will be required for renormalisability, namely those multiplied by κ_{1-5}. Each of these sets of terms is also independently $\mathcal{N}=\frac{1}{2}$-invariant. Note that, for the chiral field kinetic part of the action in Eq. (8.1), we could have combined the $U(1)$ parts with those in the $(\kappa-1)$ part of the action. We choose to leave the action in this form, however, for ease of comparison with Ref. [97].

The $\mathcal{N} = \frac{1}{2}$ transformations under which this action is invariant are given by

$$\begin{aligned}
\delta_\epsilon A_\mu^A &= -i\overline{\lambda}^A \overline{\sigma}_\mu \epsilon, \\
\delta_\epsilon \lambda_\alpha^A &= i\epsilon_\alpha D^A + (\sigma^{\mu\nu})_\alpha \left[F_{\mu\nu}^A + \frac{1}{2} i C_{\mu\nu} d^{ABC} \overline{\lambda}^B \overline{\lambda}^C \right], \\
\delta_\epsilon \overline{\lambda}_{\dot\alpha}^A &= 0, \\
\delta_\epsilon D^A &= -\epsilon \sigma^\mu D_\mu \overline{\lambda}^A, \\
\delta_\epsilon \phi &= \sqrt{2}\epsilon\psi, \\
\delta_\epsilon \overline{\phi} &= 0 \\
\delta_\epsilon \psi^\alpha &= \sqrt{2}\epsilon^\alpha F, \\
\delta_\epsilon \overline{\psi}_{\dot\alpha} &= -i\sqrt{2} \left(D_\mu \overline{\phi} \right) (\epsilon \sigma^\mu)_{\dot\alpha}, \\
\delta_\epsilon F^A &= 0, \\
\delta_\epsilon \overline{F}^A &= -i\sqrt{2} D_\mu \overline{\psi}^A \overline{\sigma}^\mu \epsilon - 2i \left(\overline{\phi} \epsilon \lambda^F \right)^A + 2C^{\mu\nu} D_\mu \left(\overline{\phi}^B \epsilon \sigma_\nu \left(\overline{\lambda}^D \right)^{AB} \right).
\end{aligned} \tag{8.6}$$

We include again the gauge-fixing term

$$S_{\text{gf}} = \frac{1}{2\alpha} \int \mathrm{d}^4 x \, (\partial \cdot A)^2, \tag{8.7}$$

which, now that we are working with a non-Abelian gauge group, will manifest itself in the Feynman diagrams as unphysical "ghost" fields.

The vector propagator is given by

$$\left(\Delta_V^{AB} \right)_{\mu\nu} = -\frac{1}{p^2} \left(\eta_{\mu\nu} + (\alpha - 1) \frac{p_\mu p_\nu}{p^2} \right) \left(e^{-1} \right)^{AB}. \tag{8.8}$$

In the chiral fields, the scalar propagator is given by

$$\Delta_\phi^{AB} = -\frac{1}{p^2} P^{AB}, \tag{8.9}$$

where

$$P^{ab} = \delta^{ab}, \qquad P^{00} = \frac{1}{\kappa}, \qquad P^{0a} = P^{a0} = 0, \tag{8.10}$$

whilst the fermion and auxiliary propagators are given by

$$\left(\Delta_\psi^{AB} \right)_{\alpha\dot\alpha} = \frac{p_\mu \sigma^\mu_{\alpha\dot\alpha}}{p^2} P^{AB} \tag{8.11}$$

and

$$\Delta_F^{AB} = P^{AB}, \tag{8.12}$$

respectively. Note that the propagators for the $U(1)$ chiral fields all now include a factor of $\frac{1}{\kappa}$.

8.3 Renormalisation of the Deformed Non-Abelian Gauge Theory

The bare $SU(N)$ fields and couplings are defined as usual by

$$
\begin{array}{ll}
\overline{\lambda}^a_B = Z_\lambda^{\frac{1}{2}}\overline{\lambda}^a, & A^a_{\mu B} = Z_A^{\frac{1}{2}} A^a_\mu \,, \\
\phi^a_B = Z_\phi^{\frac{1}{2}}\phi^a, & \overline{\phi}^a_B = Z_\phi^{\frac{1}{2}}\overline{\phi}^a \,, \\
\psi^a_B = Z_\psi^{\frac{1}{2}}\psi^a, & \overline{\psi}^a_B = Z_\psi^{\frac{1}{2}}\overline{\psi}^a \,, \\
g_B = Z_g g, & y_B = Z_y y \,, \\
C^{\mu\nu}_B = Z_C C^{\mu\nu}, & \kappa_{1-5B} = Z_{1-5} \,, \\
\multicolumn{2}{c}{(\kappa-1)_B = Z_\kappa(\kappa-1) \qquad ,}
\end{array}
\tag{8.13}
$$

whilst the $U(1)$ gauge fields and g_0 are unrenormalised. The renormalisation of the $U(1)$ chiral fields is, of course, the main topic of this chapter and so we distinguish the renormalisation constants for these fields as Z_{ϕ^0} etc. and these fields will be given a separate treatment shortly. In Eq. (8.13), Z_{1-5} are divergent contributions: that is, we have set the renormalised couplings κ_{1-5} to zero for simplicity. The other anomalous dimensions and coupling renormalisation constants are defined more usually starting with tree-level values of 1. The non-standard definition of $Z_{\kappa-1}$ is to make our results correspond more closely to those of [97]. The one-loop diagrams contributing to the undeformed terms in the action are the same as those in the $\mathcal{N}=1$ case although we must now take into account the κ-dependence of the $U(1)$ chiral field propagators as in Eqs. (8.9), (8.11), (8.12). However, the anomalous dimensions for the gauge-multiplet fields, being independent of κ, are the same as in the $\mathcal{N}=1$ theory. As our gauge-fixing term (which is the same as for the $U(1)$ case in Eq. (7.16)) does not preserve supersymmetry, the anomalous dimensions for A^a_μ and λ^a are different, as are those for ϕ^a and ψ^a. Furthermore, those for the gauge-multiplet fields are also gauge parameter-dependent. However, the gauge β-functions are gauge-independent as they must be. At one loop, we have [101, 102]

$$
\begin{aligned}
Z_\lambda &= 1 - 2g^2 NL\,(3+\alpha)\,, \\
Z_A &= 1 - g^2 NL\,(3+\alpha)\,, \\
Z_D &= 1 - 6g^2 NL, \\
Z_g &= 1 - 2g^2 NL,
\end{aligned}
\tag{8.14}
$$

where $L = \frac{1}{16\pi^2\epsilon}$.

The divergent contributions to (for instance) the scalar kinetic terms of the effective action take the form

$$
\begin{aligned}
&L\left(-\mathrm{tr}\left[\widetilde{D}^A P \widetilde{D}^B P\right] y\overline{y}\partial^\mu\overline{\phi}^A\partial_\mu\phi^B + 2g^2(1-\alpha)\,\partial^\mu\overline{\phi}^a\partial_\mu\phi^a\right) \\
&= L\left\{-y\overline{y}\left[N + \frac{4}{N\kappa}(1-\kappa)\right] + 2g^2(1-\alpha)\right\}\partial^\mu\overline{\phi}^a\partial_\mu\phi^a \\
&\quad - 2Ly\overline{y}\left[N + \frac{1}{N\kappa^2}(1-\kappa)\right]\partial^\mu\overline{\phi}^0\partial_\mu\phi^0,
\end{aligned}
\tag{8.15}
$$

which must be cancelled by

$$
-\left[Z_\phi\partial^\mu\overline{\phi}^a\partial_\mu\phi^a + Z_{\phi^0}\partial^\mu\overline{\phi}^0\partial_\mu\phi^0 + Z_\kappa(\kappa-1)\,Z_{\phi^0}\partial^\mu\overline{\phi}^0\partial_\mu\phi^0\right].
\tag{8.16}
$$

We find, using similar results for the remaining chiral kinetic terms,

$$\begin{aligned}
Z_\phi &= \left\{-y\overline{y}\left[N+\frac{4}{N\kappa}(1-\kappa)\right]+2g^2(1-\alpha)N\right\}L, \\
Z_\psi &= \left\{-y\overline{y}\left[N+\frac{4}{N\kappa}(1-\kappa)\right]-2g^2(1+\alpha)N\right\}L, \\
Z_F &= -y\overline{y}\left[N+\frac{4}{N\kappa}(1-\kappa)\right]L.
\end{aligned} \tag{8.17}$$

The assignment of the $U(1)$ chiral field renormalisation constants Z_{ϕ^0}, Z_{ψ^0}, Z_{F^0} requires more care. Consider the Yukawa term $yd^{abc}\phi^a\psi^b\psi^c$; the only diagrams contributing to this are gauge-dependent and give

$$-\frac{1}{2}(7+3\alpha)LNg^2yd^{abc}\phi^a\psi^b\psi^c. \tag{8.18}$$

We then have, at one loop,

$$\begin{aligned}
Z_y^{(1)} &= -\frac{1}{2}Z_\phi^{(1)} - Z_\psi^{(1)} - (7+3\alpha)LNg^2, \\
&= -\frac{3}{2}\left\{-y\overline{y}\left[N+\frac{4}{N\kappa}(1-\kappa)\right]+4g^2N\right\}L \\
&= -\frac{3}{2}Z_\Phi^{(1)},
\end{aligned} \tag{8.19}$$

where

$$Z_\Phi^{(1)} = \left\{-y\overline{y}\left[N+\frac{4}{N\kappa}(1-\kappa)\right]+4g^2N\right\}L \tag{8.20}$$

is the one-loop contribution to the $SU(N)$ chiral *superfield* renormalisation constant. This is in accord with the non-renormalisation theorem. The terms involving the auxilary fields in the potential require the non-linear renormalisations and so will be discussed later. In the usual case with $\kappa = 1$, the Yukawa terms involving a $U(1)$ field (for example, $\phi^0\psi^b\psi^c$) would renormalise differently from the purely $SU(N)$ terms ($\phi^a\psi^b\psi^c$) due to the difference between Z_ϕ and Z_{ϕ^0} and the different diagrams contributing to the two terms and would, therefore, require a different Yukawa coupling y' for renormalisability. In particular we would have, again from the non-renormalisation theorem,

$$Z_{y'}^{(1)} = -\frac{1}{2}Z_{\Phi^0}^{(1)} - Z_\Phi^{(1)}. \tag{8.21}$$

However, as the $\mathcal{N} = \frac{1}{2}$ transformations mix these two groups, it would seem we require them to have the same coupling. This was precisely the motivation for the introduction of κ in Ref. [97]; with this extra factor, we can adjust the value of Z_{Φ^0} to agree with Z_Φ, reducing Eq. (8.21) to

$$Z_{y'}^{(1)} = -\frac{3}{2}Z_\Phi^{(1)} = Z_y^{(1)} \tag{8.22}$$

and, thus, allowing us to identify y with y'. Also, note that the difference between Z_Φ and Z_ϕ, Z_ψ is entirely due to our choice of a non-supersymmetric gauge as is evident from the fact that the gauge-independent terms are identical. Since there are no gauge interactions for the $U(1)$ fields, this means we have

$$Z_{\phi^0} = Z_{\psi^0} = Z_{\Phi^0}. \tag{8.23}$$

	X^{ABC}	X'^{ABC}
a	$\mathrm{tr}[\widetilde{F}^A\widetilde{F}^B\widetilde{D}^C]$	0
b	0	$-\mathrm{tr}[\widehat{D}^B P\widetilde{D}^A P\widetilde{D}^C P]$

Table 8.1: Divergent contributions from Fig. 8.1

From Eqs. (8.15), (8.16), (8.20), we find

$$Z_\kappa^{(1)} = -\frac{4g^2N\kappa}{\kappa-1} + \frac{y\overline{y}N(\kappa-2)}{\kappa-1} - \frac{2y\overline{y}\left(2\kappa^2-\kappa-1\right)}{N\kappa^2}. \tag{8.24}$$

This takes care of most of the renormalisations of fields and couplings and so we now turn to those fields that require non-linear renormalisation, namely the chiral auxiliary fields and gauginos. The non-linear renormalisations of these fields are largely determined by the requirement that they cancel C-dependent divergences (other than the usual part appearing in the renormalisation of F, $\overline{F}$ in the undeformed case which follows from the results of Ref. [60]). However, we must investigate how the C-dependent divergences are modified in the presence of κ and verify that there is a consistent set of non-linear renormalisations, along with κ_{1-5} that leads to a renormalisable theory. The relevant divergent one-loop C-dependent diagrams are depicted in Figs. 8.1-8.14 using again the notation of Fig. 7.1. Fig 8.4 was, in fact, computed previously in Ref. [54] but is found to be radically different in the κ-dependent case.

The divergent contributions from Fig. 8.1 are of the form

$$\sqrt{2}C^{\mu\nu}y\overline{y}L\left(X^{ABC}\partial_\mu\overline{\phi}^A\overline{\lambda}^B\overline{\sigma}_\nu\psi^C + X'^{ABC}\overline{\phi}^A\overline{\lambda}^B\overline{\sigma}_\nu\partial_\mu\psi^C\right), \tag{8.25}$$

where the group theory factors X^{ABC}, X'^{ABC} are as given in Table 8.1 where we employ the shorthand

$$\begin{aligned}
\left(\widehat{D}^a\right)^{0b} = \left(\widehat{D}^a\right)^{b0} &= \kappa d^{ab0},\\
\left(\widehat{D}^0\right)^{00} &= \kappa d^{000},\\
\left(\widehat{D}^A\right)^{BC} &= d^{ABC} \quad \text{otherwise.}
\end{aligned} \tag{8.26}$$

Note that, although P derives from the chiral field propagators in Eqs. (8.9), (8.11), (8.12), it is redundant when surrounded by $\widetilde{F}$'s.

The divergent contributions from Fig. 8.2 are of the form

$$\sqrt{2}C^{\mu\nu}y\overline{y}LY^{ABCD}A_\mu^A\overline{\phi}^C\overline{\lambda}^B\overline{\sigma}_\nu\psi^D, \tag{8.27}$$

where Y^{ABCD} for each diagram is as given in Table 8.2.

The sum of the contributions from Figs. 8.1 and 8.2 is given by

$$\begin{aligned}
\Gamma_{1,2}^{(1)\mathrm{pole}} = y\overline{y}LC^{\mu\nu}\Big\{ &-\frac{1}{2}\left[N+\frac{8}{N\kappa}(1-\kappa)\right]d^{abc}\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu D_\mu\psi^c + \frac{N}{2}D_\mu\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^c\\
&-\left[N+\frac{4}{N\kappa}(1-\kappa)\right]d^{ab0}\left(\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu D_\mu\psi^b + \overline{\phi}^0\overline{\lambda}^a\overline{\sigma}_\nu D_\mu\psi^b\right)\\
&-\left[N+\frac{2}{N\kappa^2}\left(1-\kappa^2\right)\right]d^{ab0}\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\partial_\mu\psi^0 + Nd^{ab0}D_\mu\overline{\phi}^a\overline{\phi}^b\overline{\sigma}_\nu\psi^0\\
&-2\left[N+\frac{1}{N\kappa^2}\left(1-\kappa^2\right)\right]d^{000}\overline{\phi}^0\overline{\lambda}^0\overline{\sigma}_\nu\partial_\mu\psi^0\Big\}.
\end{aligned} \tag{8.28}$$

	Y^{ABCD}
a	$2i\mathrm{tr}[\widetilde{F}^A\widetilde{D}^B P\widetilde{D}^C P\widetilde{D}^D]$
b	$-i\mathrm{tr}[\widetilde{F}^A\widetilde{D}^B P\widetilde{D}^C P\widetilde{D}^D]$
c	$-i\mathrm{tr}[\widetilde{F}^A\widetilde{D}^D P\widetilde{D}^B P\widetilde{D}^C]$
d	0
e	$f^{ACE}\mathrm{tr}[\widetilde{F}^E\widetilde{F}^B\widetilde{D}^D]$

Table 8.2: Divergent contributions from Fig. 8.2

	Z^{ABC}	Z'^{ABC}
a	$2\mathrm{tr}[\widetilde{F}^A\widetilde{F}^B\widetilde{D}^C]$	$\frac{2}{3}\mathrm{tr}[\widetilde{F}^A\widetilde{F}^B\widetilde{D}^C] + \frac{8}{3}\mathrm{tr}[\widetilde{D}^C P\widetilde{D}^B\widetilde{D}^A P]$ $+\frac{4}{3}\mathrm{tr}[\widetilde{D}^C P\widetilde{D}^D P]d^{DAB}$
b	0	$-4\mathrm{tr}[\widetilde{D}^C P\widetilde{D}^B P\widetilde{D}^A P]$
c	$-2\mathrm{tr}[\widetilde{F}^A\widetilde{F}^B\widetilde{D}^C]$	0

Table 8.3: Divergent contributions from Fig. 8.3

The divergent contributions from Fig. 8.3 are of the form

$$iC^{\mu\nu}y\bar{y}L\left(Z^{ABC}\partial_\mu\bar{\phi}^A A^B_\nu F^C\right), \tag{8.29}$$

where Z^{ABC}, Z'^{ABC} are given as in Table 8.3. They sum to

$$\Gamma_3^{(1)\text{pole}} = iy\bar{y}LC^{\mu\nu}\left[\frac{N}{2}d^{abc}\bar{\phi}^a F^b_{\mu\nu}F^c + Nd^{ab0}\bar{\phi}^a F^b_{\mu\nu}F^0\right], \tag{8.30}$$

where we have assumed that the $\bar{\phi}AAF$ diagrams which we have not computed yield the gauge completion of the $\bar{\phi}\,(\partial A)\,F$ terms. The contributions from Fig. 8.4 are given by

$$y\bar{y}g^2LZ_1^{ABCD}\,(C\psi)^B\,\psi^A\bar{\psi}^C\bar{\psi}^D, \tag{8.31}$$

where Z_1^{ABCD} is as given in Table 8.4. These contributions sum to

$$\Gamma_4^{(1)\text{pole}} = \left[N + \frac{4}{N\kappa}(1-\kappa)\right]y\bar{y}g^2L\left[f^{abe}d^{cde}\,(C\psi)^b\,\psi^a\bar{\phi}^c\bar{\phi}^d + 2f^{abe}d^{0ce}\,(C\psi)^b\,\psi^a\bar{\phi}^c\bar{\phi}^0\right]. \tag{8.32}$$

Figs. 8.5-8.14 were computed previously in Ref. [54] and the results in the present case follow from those earlier computations with minimal modification. As such, the

	Z_1^{ABCD}
a	$-\mathrm{tr}[\widetilde{F}^C\widetilde{D}^B P\widetilde{D}^A P\widetilde{D}^D]$
b	$\mathrm{tr}[\widetilde{F}^B\widetilde{D}^C P\widetilde{D}^D\widetilde{D}^A]$
c	$-\frac{1}{3}\left(\mathrm{tr}[\widetilde{F}^B\widetilde{D}^E P\widetilde{D}^A]d^{CDE} + 2\mathrm{tr}[\widetilde{F}^B\widetilde{D}^C P\widetilde{D}^D P\widetilde{D}^A] - \mathrm{tr}[\widetilde{F}^B\widetilde{F}^C\widetilde{F}^D\widetilde{D}^A]\right)$
d	0

Table 8.4: Divergent contributions from Fig. 8.4

results are simply presented here:

$$
\begin{aligned}
\Gamma_5^{(1)\text{pole}} &= Ng^2\sqrt{2}LC^{\mu\nu}\bigg[(2+3\alpha)\,d^{abc}\partial_\mu\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^c - d^{abc}\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\partial_\mu\psi^c \\
&\quad + 2\kappa\,(1+\alpha)\,d^{ab0}\partial_\mu\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^0 - 2\kappa d^{ab0}\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\partial_\mu\psi^0 \\
&\quad + 2\alpha d^{ab0}\partial_\mu\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu\psi^b \\
&\quad + 2\,(1+\alpha)\,d^{ab0}\partial_\mu\overline{\phi}^0\overline{\lambda}^a\overline{\sigma}_\nu\psi^b\bigg], \\
\Gamma_{6,7,8}^{(1)\text{pole}} &= \sqrt{2}g^2LC^{\mu\nu}A_\mu^a\bigg[\left(\frac{7}{2}\,(1+\alpha)\,f^{bae}d^{cde} - f^{dae}d^{cbe} + \frac{1}{2}f^{bde}d^{cae}\right) \\
&\qquad \times N\overline{\phi}^b\overline{\lambda}^c\overline{\sigma}_\nu\psi^d - \frac{1}{2}\,(1+5\alpha)\,\sqrt{2N}f^{abc}\overline{\phi}^b\overline{\lambda}^0\overline{\sigma}_\nu\psi^c \\
&\quad - \frac{1}{2}\kappa\,(7+5\alpha)\,\sqrt{2N}f^{abc}\overline{\phi}^b\overline{\lambda}^c\overline{\sigma}_\nu\psi^0\bigg], \\
\Gamma_9^{(1)\text{pole}} &= iNg^2LC^{\mu\nu}\bigg[-(4-\alpha)\,d^{abc}\overline{\phi}^b\partial_\mu A_\nu^aF^c \\
&\quad - 3\kappa\,(1-\alpha)\,d^{ab0}\overline{\phi}^a\partial_\mu A_\nu^bF^0 - (5+\alpha)\,d^{ab0}\overline{\phi}^0\partial_\mu A_\nu^aF^b\bigg], \\
\Gamma_{10}^{(1)\text{pole}} &= ig^2LC^{\mu\nu}A_\mu^aA_\nu^b\bigg(\frac{1}{4}\,(3-4\alpha)\,Nf^{abc}d^{cde}\overline{\phi}^cF^d \\
&\quad - 2\alpha\kappa\sqrt{2N}f^{abc}\overline{\phi}^cF^0 + \frac{3}{2}\sqrt{2N}f^{abc}\overline{\phi}^0F^c\bigg), \\
\Gamma_{11}^{(1)\text{pole}} &= -iLNg^2C^{\alpha\beta}d^{abc}f^{cde}\overline{\phi}^a\overline{\phi}^b\psi_\alpha^c\psi_\beta^d, \\
\Gamma_{12}^{(1)\text{pole}} &= \frac{1}{2}N\overline{y}g^2LC^{\mu\nu}\,(1+\alpha)\,f^{abc}\partial_\mu\overline{\phi}^a\partial_\nu\overline{\phi}^b\overline{\phi}^c, \\
\Gamma_{13,14}^{(1)\text{pole}} &= iC^{\mu\nu}\overline{y}g^2L\bigg(-\frac{1}{2}\left(3+\frac{7}{3}\alpha\right)Nf^{abe}f^{cde}\partial_\mu\overline{\phi}^a\overline{\phi}^b\overline{\phi}^cA_\nu^d \\
&\quad + \left[-\left(\frac{5}{4}-\frac{1}{6}\alpha\right)Nd^{abe}d^{cde} + \left(3+\frac{7}{3}\alpha\right)\delta^{ab}\delta^{cd}\right]\overline{\phi}^a\overline{\phi}^b\overline{\phi}^c\partial_\mu A_\nu^d \\
&\quad - \frac{1}{2}\,(9+\alpha)\,\sqrt{2N}d^{abc}\overline{\phi}^0\overline{\phi}^a\overline{\phi}^b\partial_\mu A_\nu^c - (5+\alpha)\,\overline{\phi}^0\overline{\phi}^0\overline{\phi}^a\partial_\mu A_\nu^a \\
&\quad - 2\sqrt{2N}d^{abc}\overline{\phi}^a\overline{\phi}^b\overline{\phi}^c\partial_\mu A_\nu^0 - 8\overline{\phi}^a\overline{\phi}^a\overline{\phi}^0\partial_\mu A_\nu^0\bigg).
\end{aligned}
\tag{8.33}
$$

At this point we turn to the remaining renormalisations, the non-linear renormalisations of F, $\overline{F}$ and λ, required to cancel the divergences. The renormalisation of λ^A is given by

$$
\begin{aligned}
\lambda_B^a &= Z_\lambda^{\frac{1}{2}}\lambda^a - \frac{1}{2}NLg^2C^{\mu\nu}d^{abc}\sigma_\mu\overline{\lambda}^cA_\nu^b - NLg^2C^{\mu\nu}d^{ab0}\sigma_\mu\overline{\lambda}^0A_\nu^b \\
&\quad + i\sqrt{2}\tau_1NLg^4d^{abc}\,(C\psi)^b\,\overline{\phi}^c + i\sqrt{2}\tau_2NLg^4d^{ab0}\,(C\psi)^0\,\overline{\phi}^b, \\
\lambda_B^0 &= \lambda^0 + i\sqrt{2}\tau_3NLg^2g_0^2d^{0ab}\,(C\psi)^a\,\overline{\phi}^b,
\end{aligned}
\tag{8.34}
$$

where $(C\psi)^\alpha = C^\alpha{}_\beta\psi^\beta$. The coefficients of the non-linear terms on the first line of Eq. (8.34) were computed previously in Ref. [51]. The new terms involving τ_{1-3} will be returned to later. Replacing λ by λ_B produces the following change in the action:

$$
\begin{aligned}
S_0\left(\lambda_B\right)-S_0\left(\lambda\right)=NLg^2\int \mathrm{d}^4x\Big\{ & -\frac{1}{2}f^{bde}d^{cae}A^a_\mu\overline{\phi}^b\overline{\lambda}^c\overline{\sigma}_\nu\psi^d - f^{abe}d^{ec0}A^a_\mu\overline{\phi}^b\overline{\lambda}^0\overline{\sigma}_\nu\psi^c \\
& + \tau_1\big[ig^2d^{abe}f^{cde}\overline{\phi}^a\overline{\phi}^b\psi^c\left(C\psi^d\right) \\
& + \sqrt{2}C^{\mu\nu}d^{abc}\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu D_\mu\psi^c + \sqrt{2}C^{\mu\nu}d^{abc}D_\mu\overline{\phi}^a\overline{\phi}^b\overline{\sigma}_\nu\psi^0\big] \\
& + \tau_2\sqrt{2}C^{\mu\nu}d^{ab0}\left(\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\partial_\mu\psi^0 + D_\mu\overline{\phi}^a\overline{\lambda}^b\overline{\sigma}_\nu\psi^0\right) \\
& + \tau_3\sqrt{2}C^{\mu\nu}d^{0ab}\left(\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu D_\mu\psi^b + D_\mu\overline{\phi}^a\overline{\lambda}^0\overline{\sigma}_\nu\psi^b\right) \\
& + \left(\text{pure gauge terms}\right)\Big\},
\end{aligned}
\tag{8.35}
$$

where the pure gauge terms are given in Ref. [51].

In order for the contributions linear in F to be finite, we also require

$$
\begin{aligned}
\overline{F}^a_B &= Z_F\overline{F}^a + iC^{\mu\nu}Lg^2\Big\{N\left[\left(5+2\alpha\right)\partial_\mu A^b_\nu - \frac{1}{4}\left(11+4\alpha\right)f^{bde}A^d_\mu A^e_\nu\right]\overline{\phi}^c d^{abc} \\
&\quad + \sqrt{2N}\left[2\left(2+\alpha\right)\partial_\mu A^a_\nu - \frac{1}{2}\left(5+2\alpha\right)f^{abc}A^b_\mu A^c_\nu\right]\overline{\phi}^0 \\
&\quad + 2\sqrt{2N}\left(3+\alpha\right)\partial_\mu A^0_\nu\overline{\phi}^a\Big\} + \left(\alpha+3\right)g^2NL\frac{1}{4}yd^{abc}\phi^b\phi^c \\
&\quad + \frac{1}{2}\left(\alpha+3\right)yg^2NLd^{ab0}\phi^b\phi^0 + \tau_4 g^2yLf^{abc}\left(C\psi\right)^b\psi^c + \ldots, \\
\overline{F}^0_B &= Z_F\overline{F}^0, \\
F^a_B &= Z_FF^a + \left(\alpha+3\right)g^2NL\frac{1}{4}\overline{y}d^{abc}\overline{\phi}^b\overline{\phi}^c + \frac{1}{2}\left(\alpha+3\right)\overline{y}g^2NLd^{ab0}\overline{\phi}^b\overline{\phi}^0 + \ldots, \\
F^0_B &= Z_FF^0,
\end{aligned}
\tag{8.36}
$$

where the ellipsis stands for $\overline{\phi\lambda\lambda}$ terms which only affect the separately $\mathcal{N}=\frac{1}{2}$-independent terms which we chose to omit. Setting

$$
Z^{(1)}_n = z_nL,
\tag{8.37}
$$

we find that with the following choices:

$$
\begin{aligned}
\tau_1 &= 1, \\
\tau_2 &= -2, \\
\tau_3 &= 4, \\
\tau_4 &= \left[N + \frac{2}{N\kappa^2}\left(1-\kappa^2\right)\right], \\
z_C &= 0, \\
z_1 &= \frac{1}{2}\left[N + \frac{8}{N\kappa}\left(1-\kappa\right)\right]y\overline{y}, \\
z_2 &= \left[N + \frac{4}{N\kappa}\left(1-\kappa\right)\right]y\overline{y}, \\
z_3 &= \left[N + \frac{2}{N\kappa^2}\left(1-\kappa^2\right)\right]y\overline{y} + 4g^2,
\end{aligned}
\tag{8.38}
$$

$$z_4 = \left[N + \frac{4}{N\kappa}(1-\kappa)\right] y\bar{y} - 4g^2,$$
$$z_5 = 2\left[N + \frac{1}{N\kappa^2}(1-\kappa)\right] y\bar{y},$$

the one-loop effective action is finite. In particular, the same coupling y is sufficient for the renormalisation of the full set of potential terms allowing for a renormalisable non-Abelian theory. An additional complication that was present in the case of Ref. [54] was the necessity of introducing extra non-anticommutativity parameters $C_{1,2}$ to allow for the different renormalisations of non-anticommutative terms. The form of the action arrived at by the introduction of κ, however, allows for a single non-anticommutativity parameter, $C^{\mu\nu}$, which is unrenormalised at one-loop. The groups of terms involving κ_{1-5} have an analogue in Ref. [97] in the groups of terms involving t_{1-5}, each group being separately invariant. Explicit one-loop results are not given there for t_{1-5} but it is unlikely that we would have agreement anyway due to our different choices of gauge.

8.4 Summary

Reformulating the work of [97] in terms of component fields, we have confirmed their conclusion that we can construct a renormalisable non-anticommutative $SU(N)\otimes U(1)$ theory with a superpotential by means of a rescaling of the kinetic term of the $U(1)$ fields. This allows for a single Yukawa coupling for all chiral superfields thereby avoiding the problems discussed at the beginning of this chapter and solving the difficulties of Ref. [54]. In addition to restoring renormalisability, it allows for a single non-anticommutativity tensor which is unrenormalised at one loop.

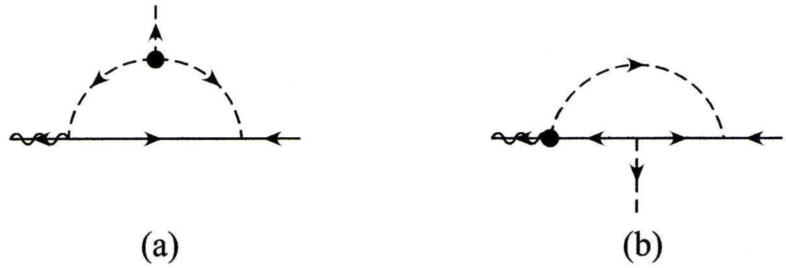


Figure 8.1: Diagrams with one gaugino, one scalar and one chiral fermion line (and two Yukawa couplings)

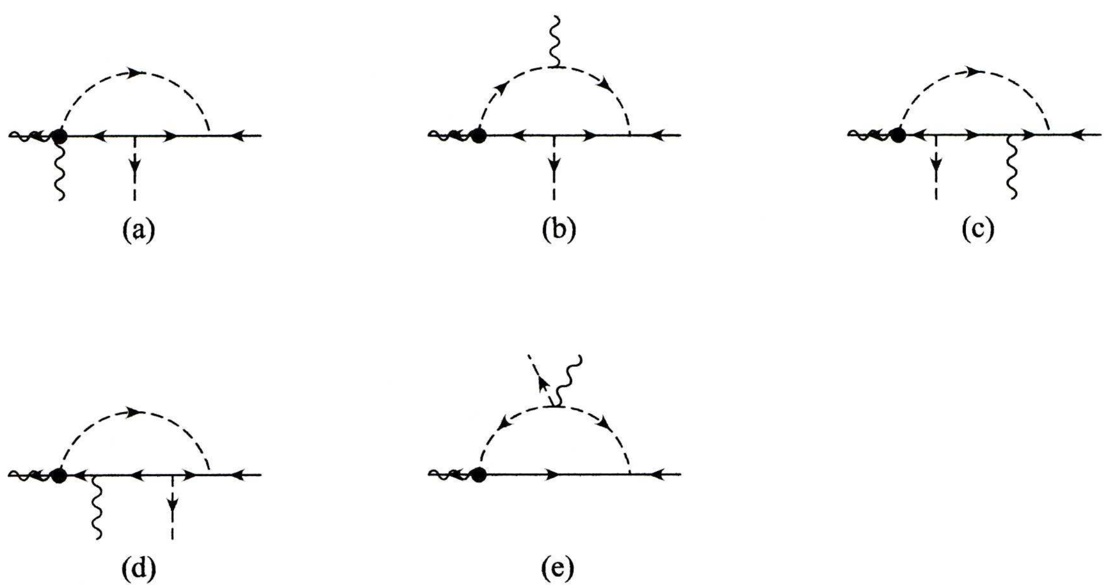


Figure 8.2: Diagrams with one gaugino, one scalar, one chiral fermion and one gauge line (and two Yukawa couplings)

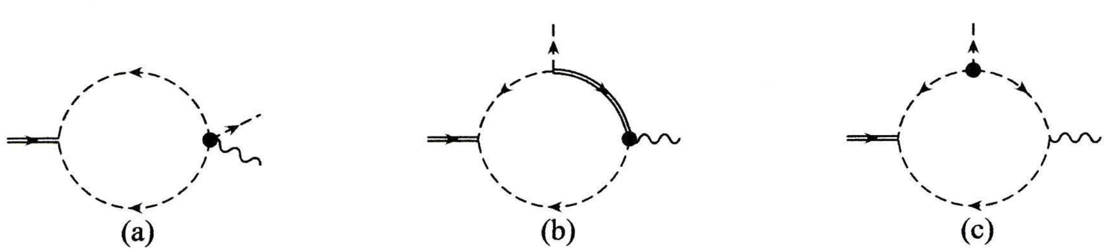


Figure 8.3: Diagrams with one auxiliary, one scalar and one gauge line (and two Yukawa couplings)

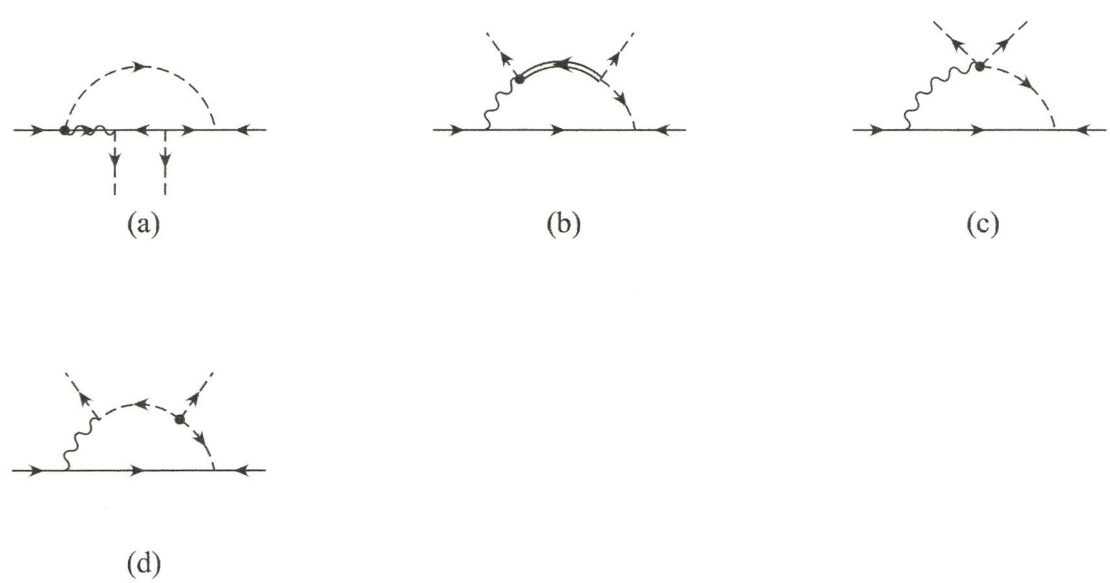


Figure 8.4: Diagrams with two chiral fermion lines and two scalars (and two Yukawa couplings)

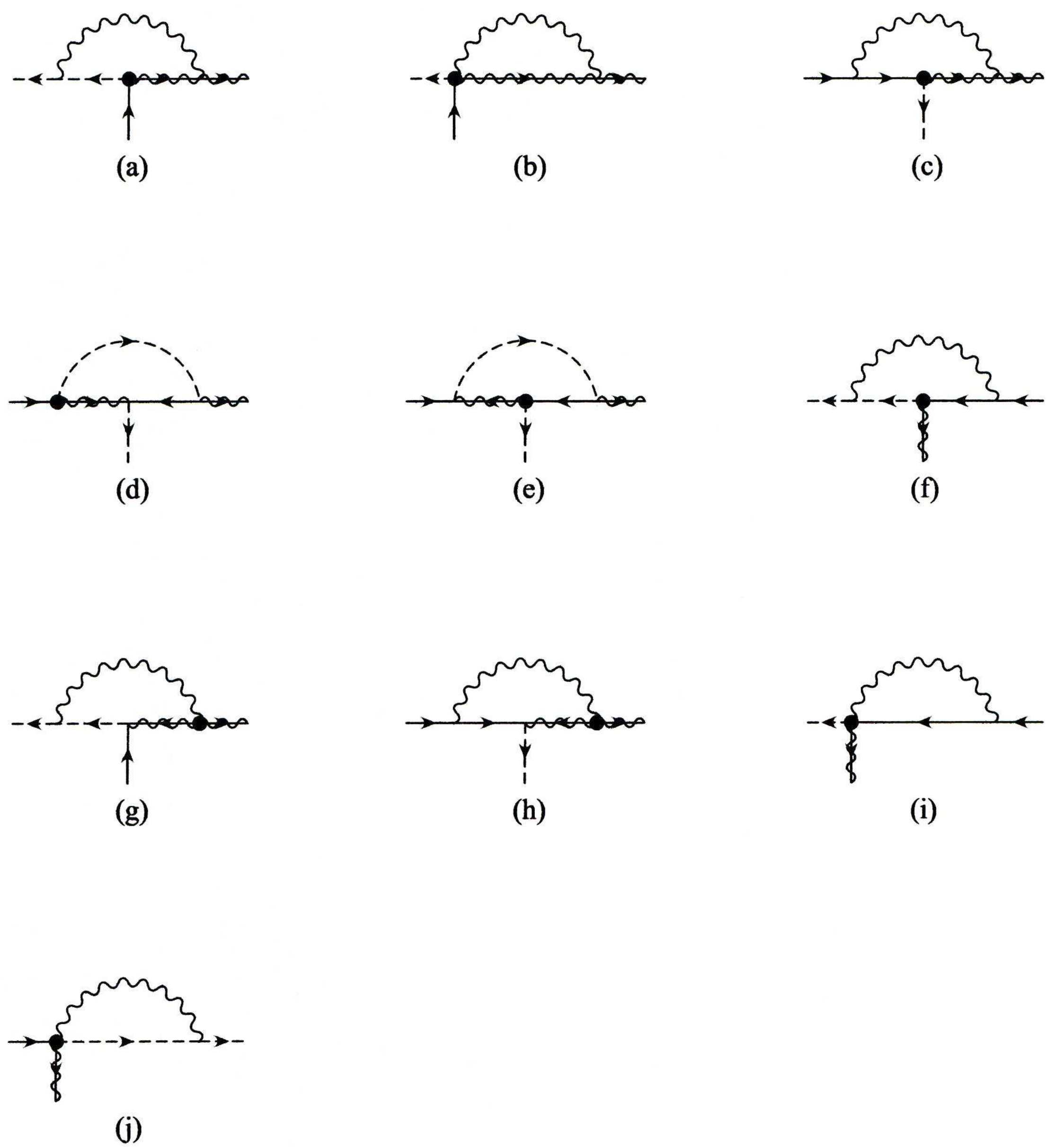


Figure 8.5: Diagrams with one gaugino, one scalar and one chiral fermion line

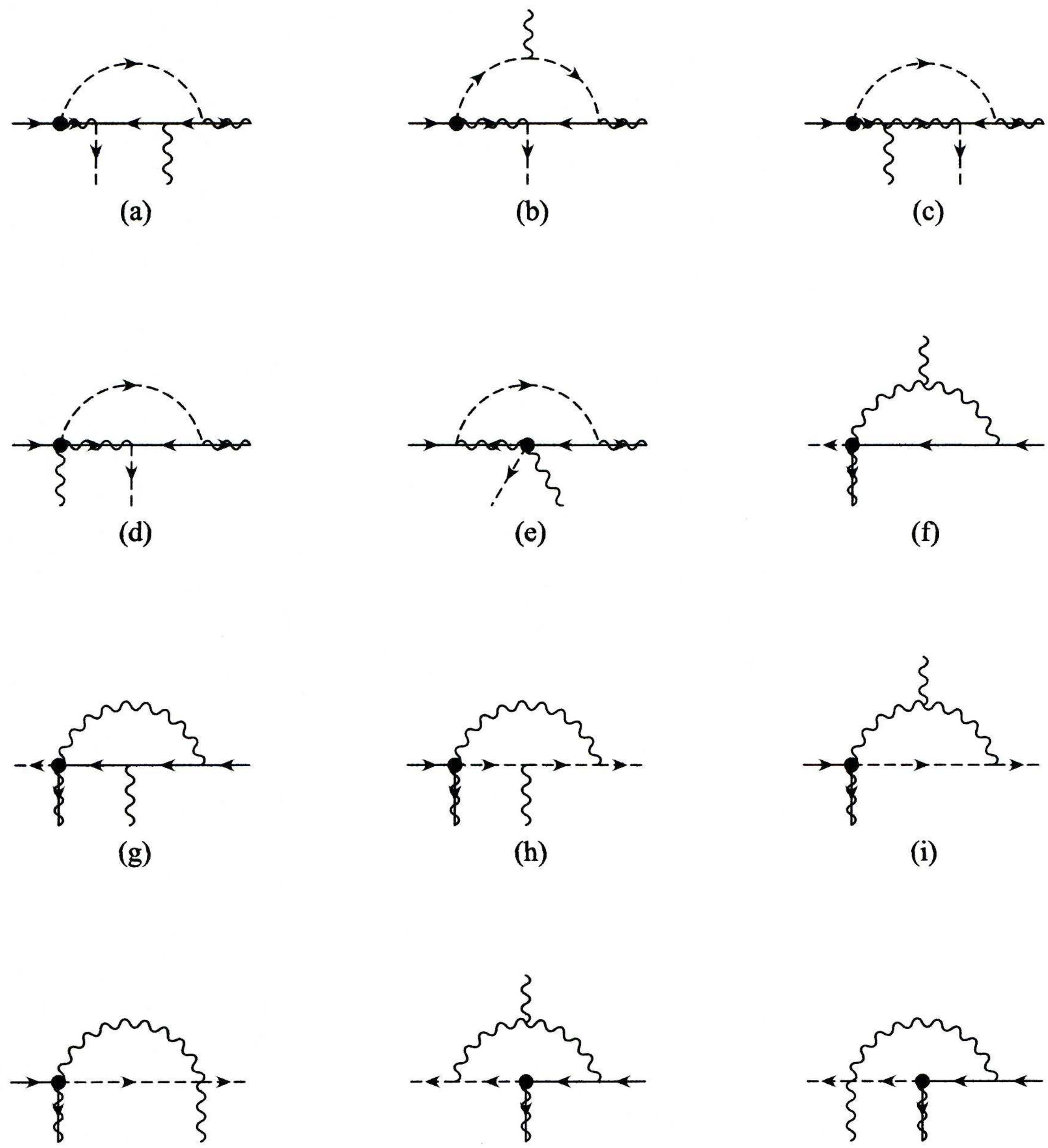


Figure 8.6: Diagrams with one gaugino, one scalar, one chiral fermion and one gauge line

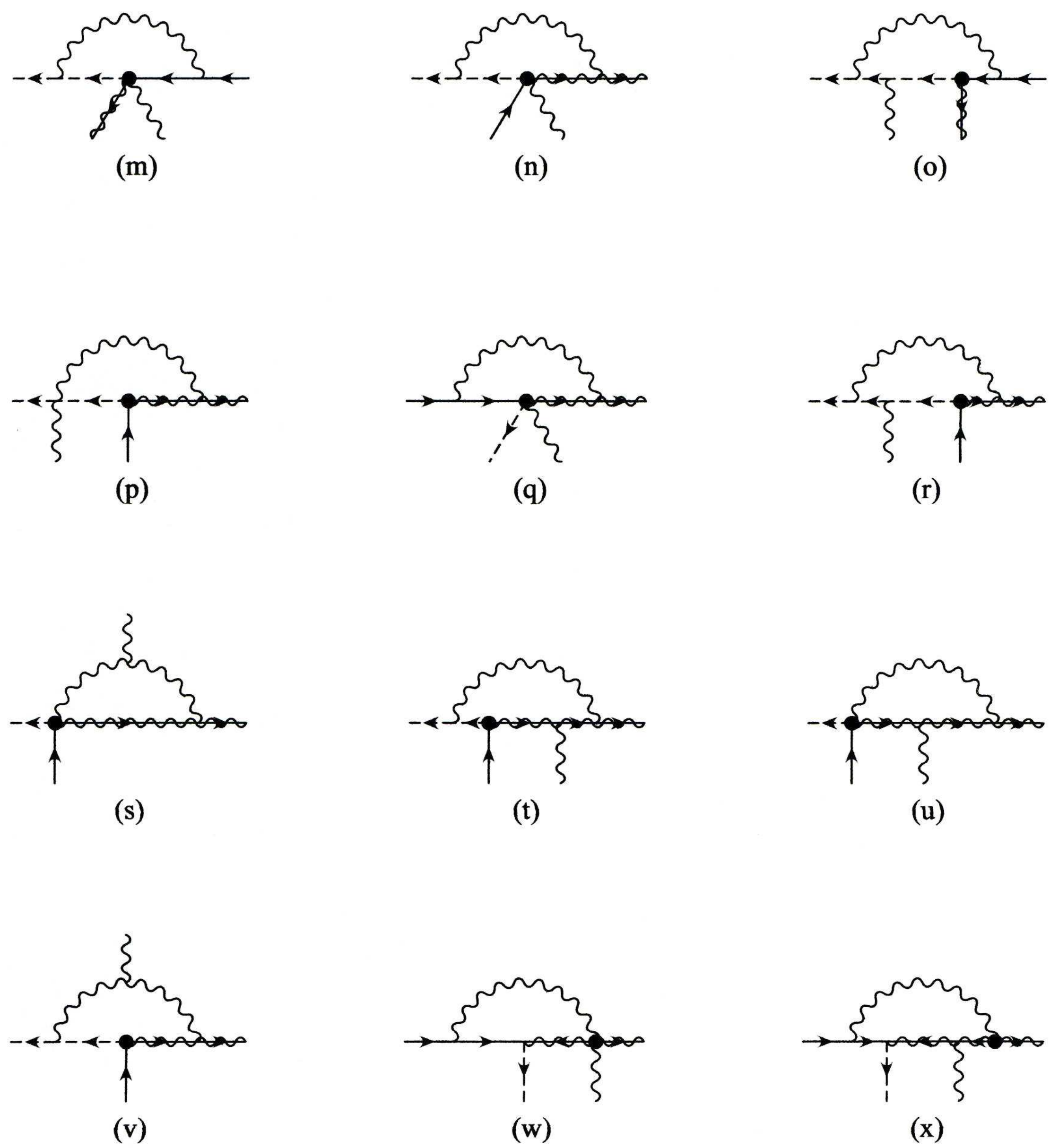


Figure 8.7: Diagrams with one gaugino, one scalar, one chiral fermion and one gauge line (continued)

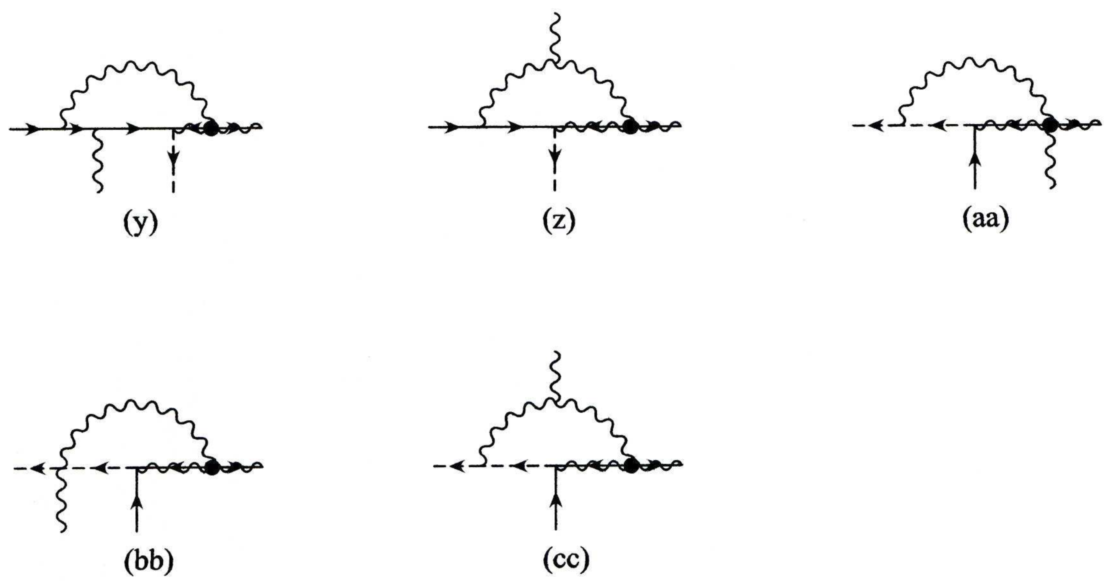


Figure 8.8: Diagrams with one gaugino, one scalar, one chiral fermion and one gauge line (continued)

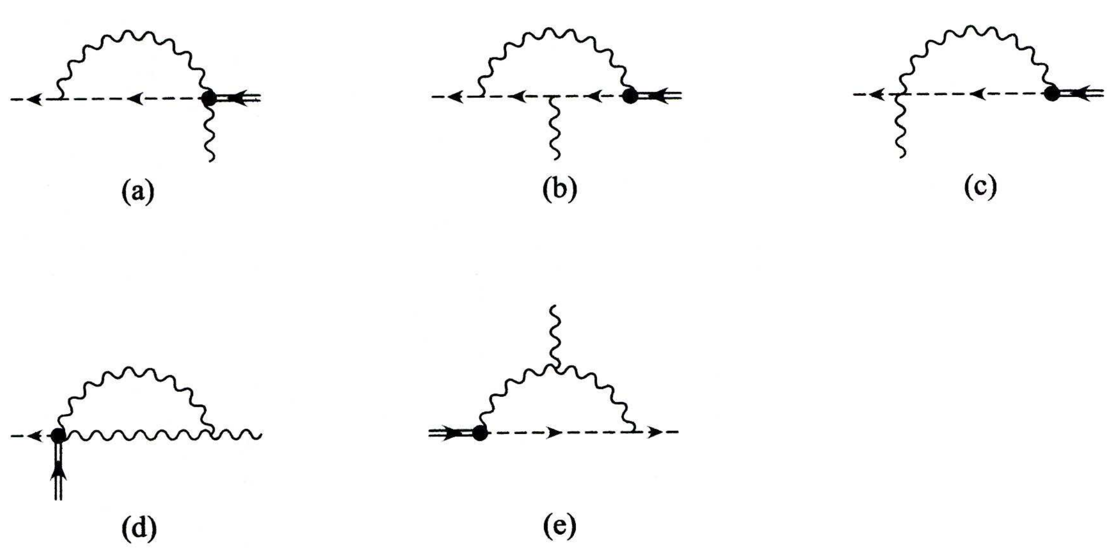


Figure 8.9: Diagrams with one gauge, one scalar and one auxiliary line

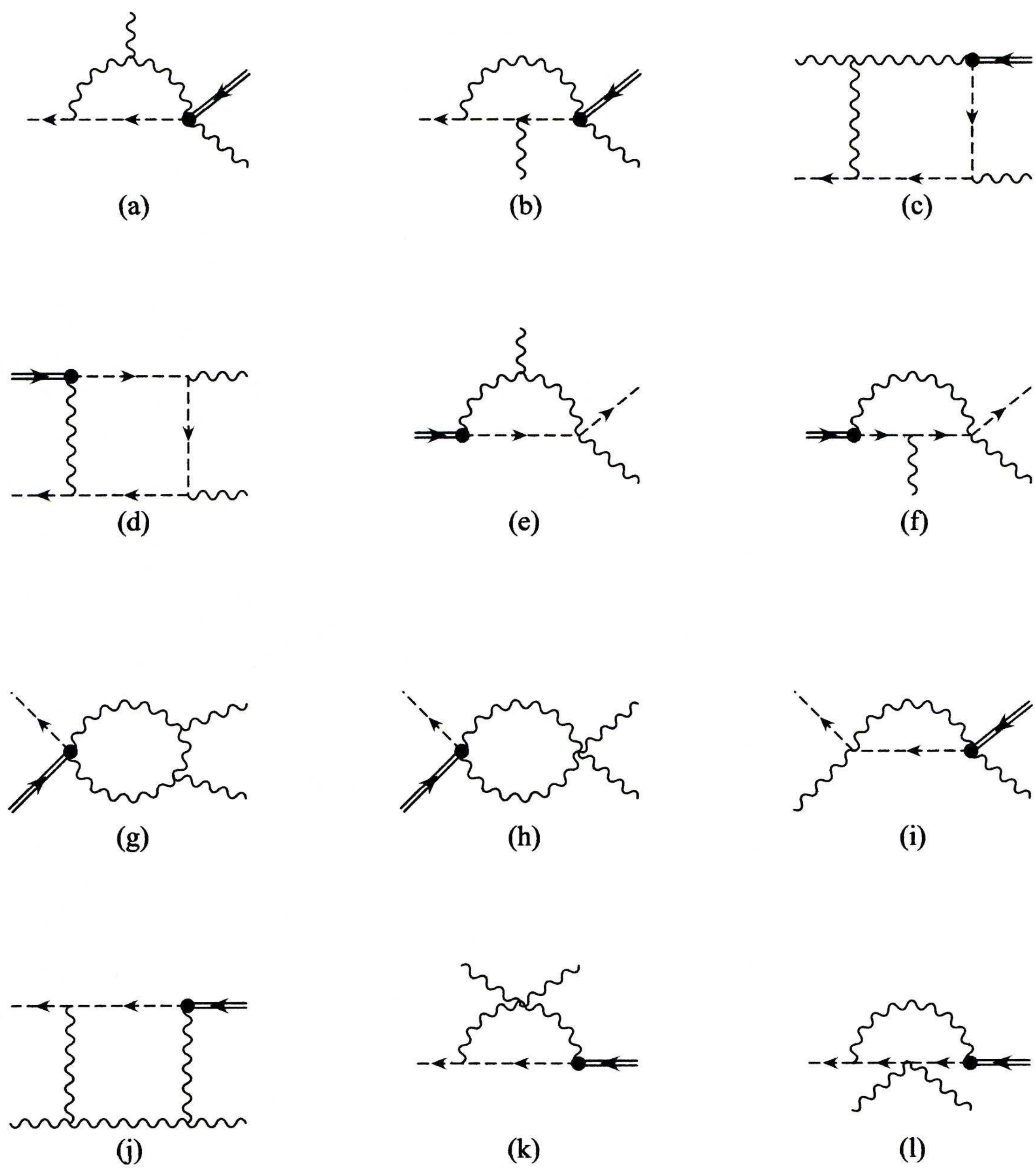


Figure 8.10: Diagrams with two gauge, one scalar and one auxiliary line

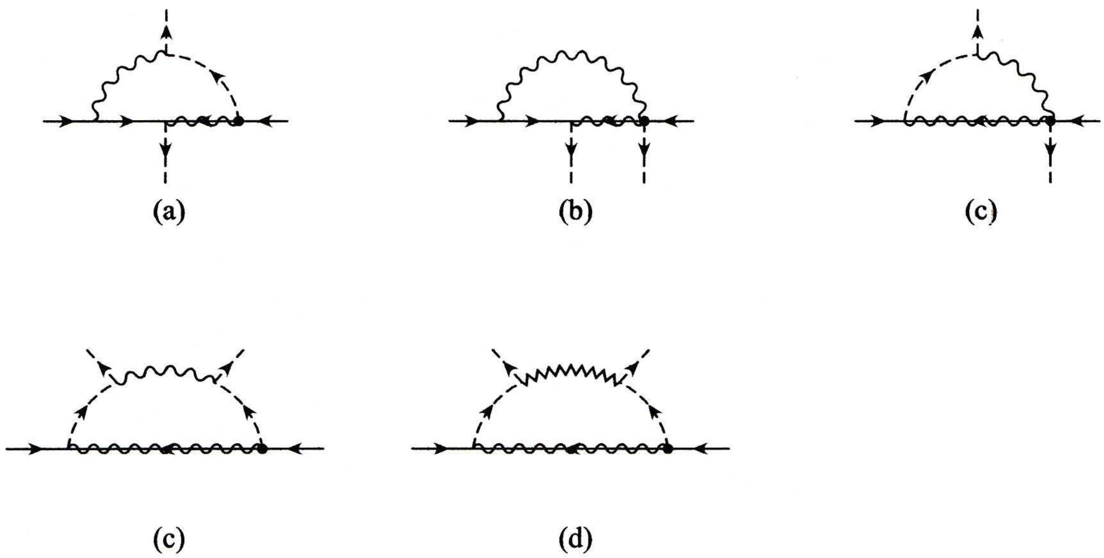


Figure 8.11: Diagrams with two scalar and two chiral fermion lines

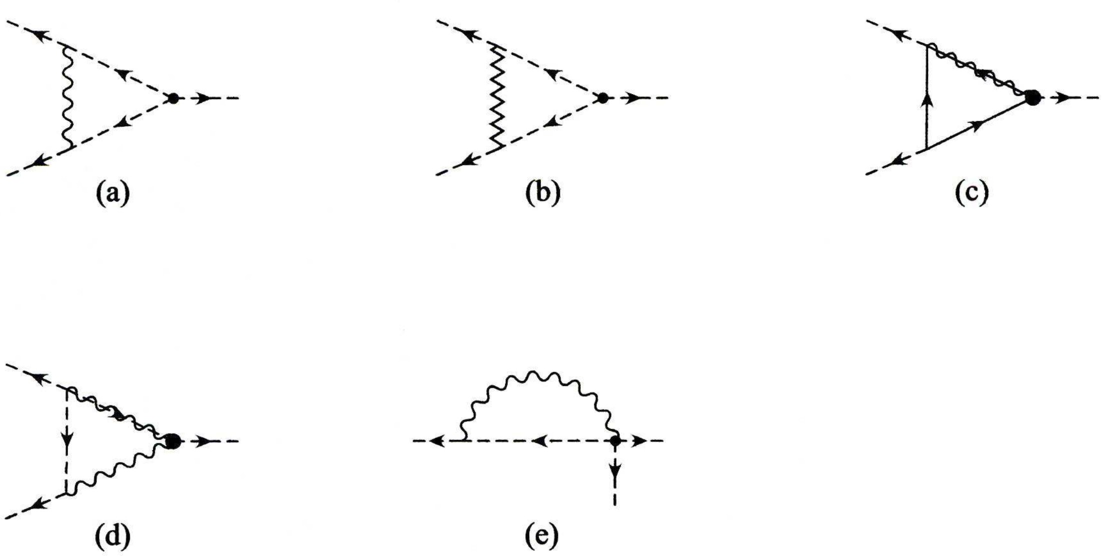


Figure 8.12: Diagrams with three scalar lines

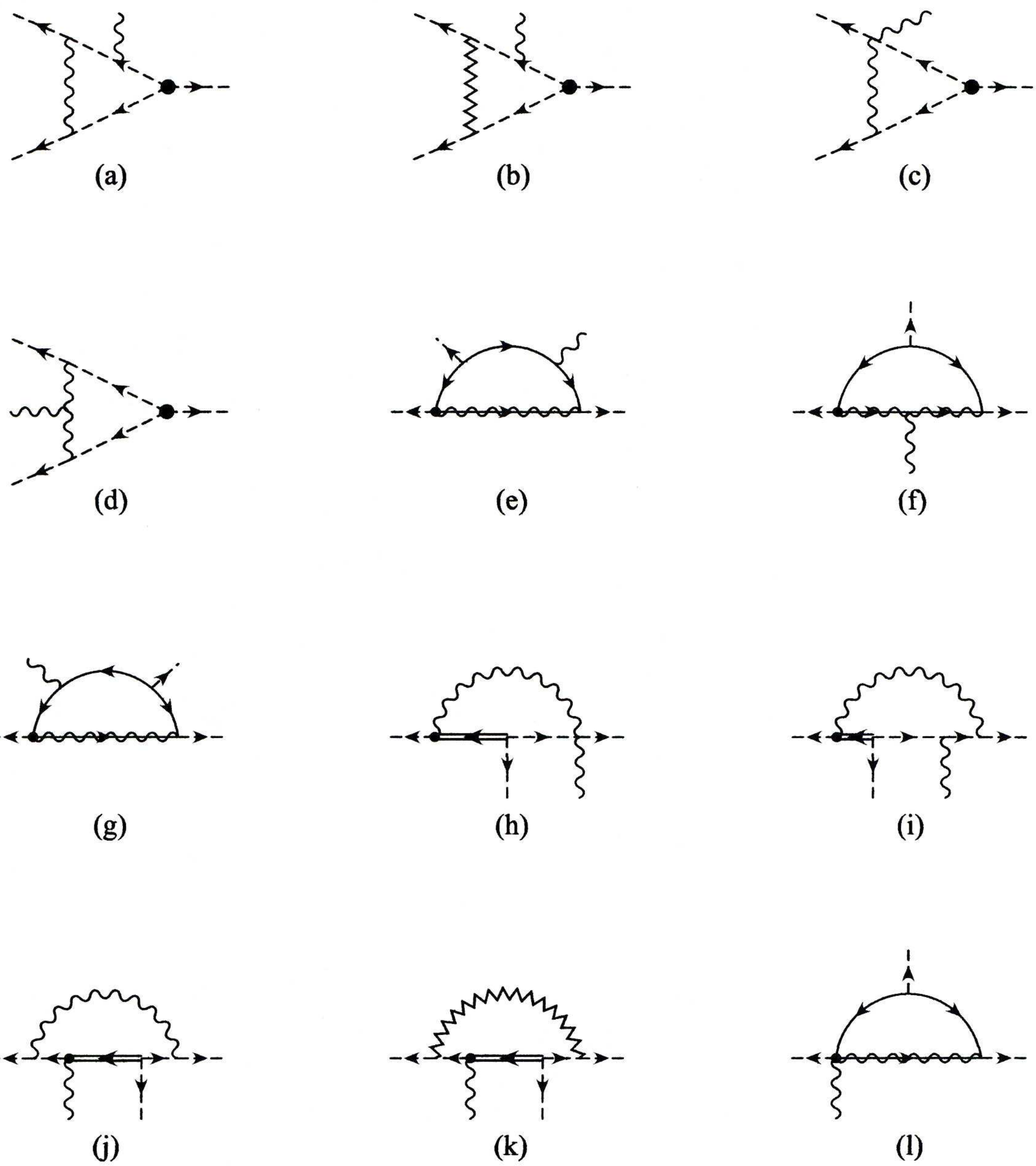


Figure 8.13: Diagrams with three scalar, one gauge line

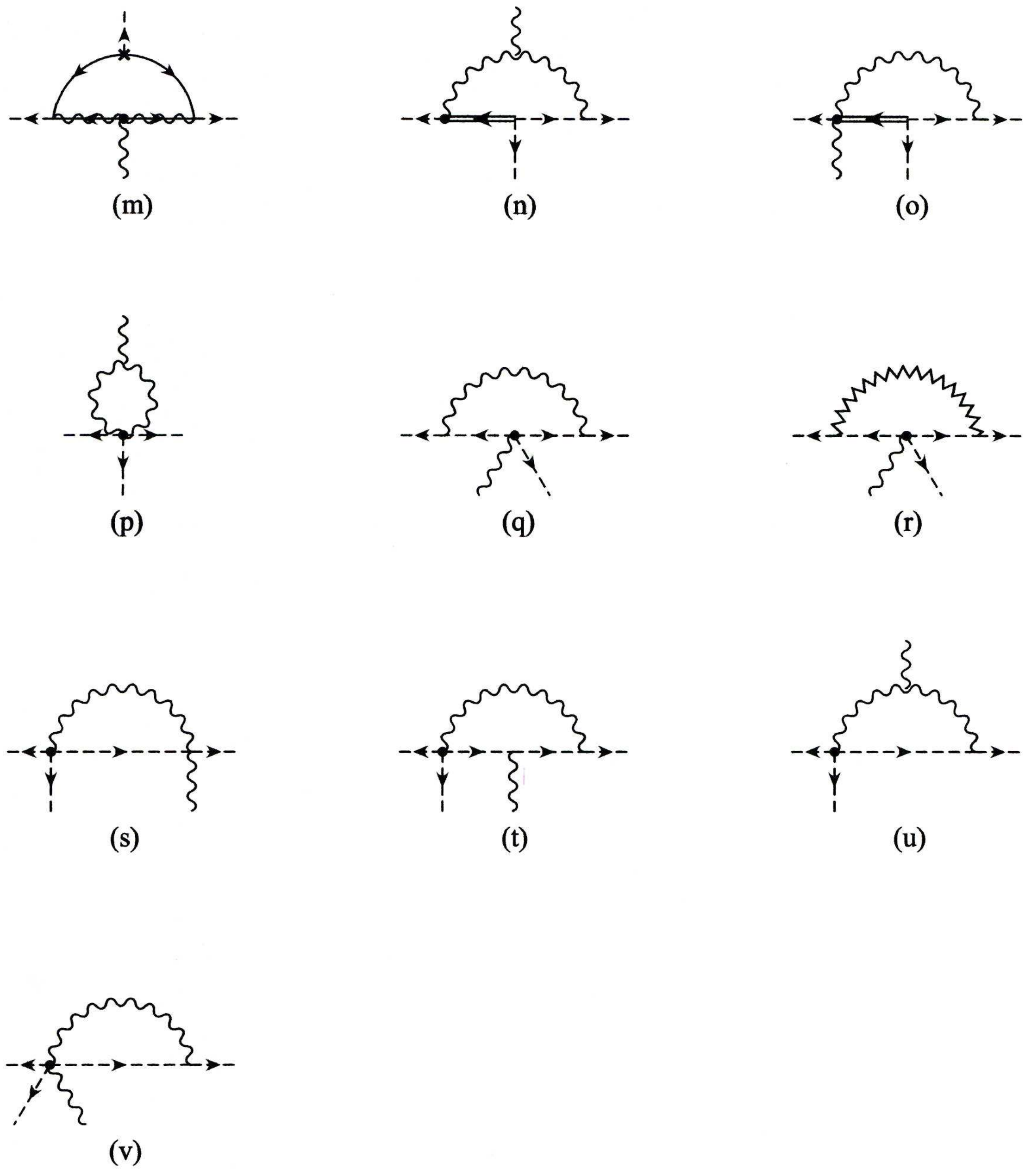


Figure 8.14: Diagrams with three scalar, one gauge line (continued)

Chapter 9

Conclusion

Although non-anticommutative theories were originally studied as a particular limit of String Theory, the study of such theories in the more general setting demonstrates some of the intricacies of the renormalisation process. We have discussed the renormalisation of four different non-anticommutative supersymmetric theories. The importance of symmetries in the Lagrangian for renormalisability was made clear by the fact that the two-dimensional non-linear σ-model is non-renormalisable in the non-anticommutative case. This is despite the fact that the undeformed model is renormalisable, even with arbitrarily high interaction terms, due to the zero mass dimension of the fields and also, therefore, of any couplings. However, the deformation of superspace introduces terms with a (negative) dimensionful coupling $C^{\mu\nu}$, and the σ-model is then found to be non-renormalisable. The undeformed four-dimensional Wess-Zumino model, on the other hand, permits only cubic terms in its superpotential for renormalisability, which appears to be more restrictive than the two-dimensional case. Despite this, the introduction of a deformation does not destroy the renormalisability of the theory, as we have seen from calculating the divergent contributions arising from one- and two-loop diagrams and their subsequent absorption into the parameters of the theory. This renormalisability is thanks to non-Hermiticity and the presence of an R-symmetry in the classical action. For form-invariance, however, it does require the introduction of new (deformed) terms in the action, which can absorb the divergences arising from the original deformation. Four natural forms for the action, requiring no non-linear renormalisation of the auxiliary fields, were discovered at one loop, and two of these were shown to apply also in the two-loop case.

In much the same way that gauge theories may be extended to include supersymmetry with minimal modifications, as was discussed in §2.4, supersymmetric gauge theory may also be extended to the non-anticommutative case with the introduction of only a few new elements, such as non-linear renormalisation of gauginos. The form that a gauge theory can take in non-anticommutative superspace is restricted by considerations of renormalisability and self-consistency, and it would appear at first that a non-Abelian gauge symmetry forbids the introduction of a superpotential. However, as we have shown through a complete one-loop analysis, a theory with both an *Abelian* gauge symmetry and a superpotential is found to be renormalisable. Natural forms for this action were also found which were related to those in the ungauged case. In addition, thanks to the work of [97], another (minor) modification of the action allows for a non-Abelian theory with a potential, a conclusion which we have confirmed in the component formalism. One of the reasons for the introduction of supersymmetry was the hope of finding power-counting non-renormalisable theories that were, in fact, renormalisable. The non-anticommutative theories presented in the previous three chapters provide concrete examples of theories with this property.

The renormalisability of the $SU(N) \otimes U(1)$ theory has revived the possibility that the C-parameter is unrenormalised at one loop, as it appeared to be in Refs. [50, 51]. More recent work aims to show that this behaviour extends to higher orders. As an investigation of the full two-loop behaviour of a deformed theory would be rather complex, work so far has concentrated on partial, more manageable calculations. As the parameter must renormalise in the same way in each term of the action (for $\mathcal{N} = \frac{1}{2}$ invariance), it is sufficient to investigate the renormalisation in a single term. With this in mind, the first calculation is the two-loop renormalisation of the $\overline{y}C^{\mu\nu}f^{abc}\partial_\mu\overline{\phi}^a\partial_\nu\overline{\phi}^b\overline{\phi}^c$ term of the $SU(N)\otimes U(1)$ theory of the previous chapter. The reasons for this particular choice of interaction are two-fold. Firstly, the interaction contains no auxiliary field and is not affected by the non-linear renormalisation of F discussed in previous chapters. This eliminates from the calculation any ambiguities relating to these non-linear renormalisations, such as those discussed in chapter 6. Secondly, there are difficulties arising in the computation of fermion diagrams at two loops due to the appearance of factors such as $\epsilon^{\mu\nu\rho\tau}\sigma_{\rho\tau}$, for which identities are ambiguous when evaluated away from four dimensions. The interaction term was chosen so as to contain no external fermion lines in order to reduce the number of occurrences of such ambiguous terms; in fact, our choice proved to avoid such factors altogether. However, a similar problem arose in the form of an $\epsilon^{\mu\nu\rho\sigma}C_{\rho\sigma}$ factor. This factor is easily evaluated in four-dimensions, giving $\epsilon^{\mu\nu\rho\sigma}C_{\rho\sigma} \equiv 2C^{\mu\nu}$, as C is self-dual. A modification of this relation away from four dimensions (such as exists for $\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$) would seem a reasonable assumption; however, consistent renormalisation appears to impose the four-dimensional relation in the general case. The consequence of this is that we must modify instead the relation $C^{\mu\nu} = C^{\alpha\beta}\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha^{\ \gamma}$ away from four dimensions.

Working at higher loop orders has returned us to the issue raised in §4.2.3 of the differences between dimensional regularisation (DREG) and its supersymmetric counterpart, dimensional reduction (DRED). The two-loop calculation of Z_ϕ included in this calculation was computed (in Ref. [103]) in DREG and must, therefore, be translated into DRED. A further proviso for this calculation was the choice to limit the considered contributions to those containing a Yukawa vertex. This choice was initially made simply to avoid a proliferation of contributions since the remaining $O(g^4)$ diagrams are numerous. Therefore, any conclusion drawn from the calculation is only correct up to quartic order in the gauge coupling. However, this calculation does confirm that the C-parameter has no Yukawa-dependent renormalisation at two loops which seems to support our conjecture that C remains unrenormalised at this order [104].

Of course, it would be preferable to include the g^4 terms in the calculation in order to provide a conclusive result for the two-loop renormalisation of C. Unfortunately, there is an additional complication concerning these diagrams: namely the appearance of contractions of the form $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta}$, whose consistency with identities involving C is not clear away from four dimensions. One possibility for circumventing these complications is the adoption of a different regularisation scheme: differential regularisation allows one to work purely in four dimensions and would, therefore, avoid the ambiguities mentioned above [105]. Another approach is to avoid these terms by restricting calculations to the ungauged theory. For this reason, another avenue of inquiry is into the higher-order behaviour of the deformed Wess-Zumino model of chapter 6. In particular, we are investigating the three-loop behaviour of the $\overline{y}C^{\mu\nu}\overline{\phi}\partial_\mu\overline{\phi}\partial_\nu\overline{\phi}$ term in this model — a calculation which is currently still in progress.

An alternative means of sidestepping the problematic terms mentioned above would be to avoid loop calculations altogether. It is possible that one may be able to find a Slavnov-Taylor–type identity guaranteeing the non-renormalisation of the C-parameter to all orders. The most natural approach to finding such an identity would seem to be via superfields. Preliminary efforts in this direction in the superfield formalism appear promising but it is, as yet, unclear how the result will manifest itself in components.

Appendix A

Useful Formulae for Dimensional Regularisation

There follows a list of formulae for d-dimensional momentum space integrals in Euclidean space

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$
$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1}}{(4\pi)^{\frac{d}{2}}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$
$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \times \frac{1}{4} (\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \tag{A.1}$$

from which we may derive the following

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2)^m[(k-p)^2]^n} = \frac{1}{16\pi^2} \frac{\Gamma(m+n-\frac{d}{2})}{\Gamma(m)\Gamma(n)} \frac{\Gamma(\frac{d}{2}-n)\Gamma(\frac{d}{2}-m)}{\Gamma(d-m-n)} \frac{1}{(p^2)^{m+n-\frac{d}{2}}}$$
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu}{(k^2)^m[(k-p)^2]^n} = \frac{1}{16\pi^2} \frac{\Gamma(m+n-\frac{d}{2})}{\Gamma(m)\Gamma(n)} \frac{\Gamma(\frac{d}{2}-n)\Gamma(\frac{d}{2}-m+1)}{\Gamma(d-m-n+1)} \frac{p_\mu}{(p^2)^{m+n-\frac{d}{2}}}$$
$$\begin{aligned}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2)^m[(k-p)^2]^n} = {}& \frac{1}{16\pi^2} \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{\Gamma(d-m-n+2)} \frac{1}{(p^2)^{m+n-\frac{d}{2}}} \times \\ & \times \left[\frac{1}{2}p^2\delta_{\mu\nu}\Gamma(m+n-\frac{d}{2}-1)\Gamma(\frac{d}{2}-m+1)\Gamma(\frac{d}{2}-n+1) \right. \\ & \left. + p_\mu p_\nu \Gamma(m+n-\frac{d}{2}\Gamma(\frac{d}{2}-m+2)\Gamma(\frac{d}{2}-n)\right]\end{aligned} \tag{A.2}$$

by means of

$$\frac{1}{A_1^m A_2^n} = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \delta(x_1 + x_2 - 1) \frac{x_1^m x_2^n}{(x_1 A_1 + x_2 A_2)^{m+n}} \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \tag{A.3}$$

The Γ function has poles at every non-positive integer. Around these points, it can be expanded as

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!\epsilon} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathcal{O}(\epsilon) \tag{A.4}$$

where γ is the Euler-Mascheroni constant.

In addition, $\Gamma(x)$ obeys

$$\Gamma(1+x) = x\Gamma(x) \ \forall x \tag{A.5}$$

Appendix B

Identities for $SU(N)$

The following identities are of use in computing group theory factors [100]

$$\begin{aligned}
&\operatorname{tr}\left[\widetilde{D}^a\widetilde{D}^b\right] = \frac{N^2-4}{N}\delta^{ab}, \quad \operatorname{tr}\left[\widetilde{D}^a\widetilde{D}^b\widetilde{D}^c\right] = \frac{N^2-12}{2N}d^{abc}, \\
&\operatorname{tr}\left[\widetilde{F}^a\widetilde{F}^b\widetilde{D}^c\right] = \frac{N}{2}d^{abc}, \qquad \operatorname{tr}\left[\widetilde{F}^a\widetilde{D}^b\widetilde{D}^c\right] = i\frac{N^2-4}{2N}f^{abc}, \\
&\operatorname{tr}\left[\widetilde{F}^a\widetilde{F}^b\widetilde{F}^c\widetilde{D}^d\right] = i\frac{N}{4}\left(d^{abx}f^{cdx} + d^{cdx}f^{abx}\right), \\
&\operatorname{tr}\left[\widetilde{F}^a\widetilde{D}^b\widetilde{D}^c\widetilde{D}^d\right] = \frac{N^2-12}{4N}f^{abx}d^{cdx} + \frac{N}{4}d^{abx}f^{cdx} \\
&\qquad + \frac{1}{N}\left(f^{adx}d^{cbx} - f^{acx}d^{bdx}\right)
\end{aligned} \tag{B.1}$$

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